

# On the Location of Steiner Points in Uniformly-Oriented Steiner Trees\*

Benny K. Nielsen<sup>†</sup>   Pawel Winter<sup>†</sup>   Martin Zachariasen<sup>†</sup>

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## Abstract

We give a fundamental result on the location of Steiner points for Steiner minimum trees in uniform orientation metrics. As a corollary we obtain a linear time algorithm for constructing a Steiner minimum tree for a given full topology when the number of uniform orientations is  $\lambda = 3m$ ,  $m \geq 1$ .

**Keywords:** computational geometry, interconnection networks, Steiner trees

## 1 Introduction

The *Steiner tree problem* asks for a shortest possible network interconnecting a given set of points. Unlike the minimum spanning tree problem, three or more edges of *Steiner minimum trees (SMTs)* can meet anywhere and not only at the given points. These additional junctions are usually referred to as *Steiner points*.

There are two very important variants of the geometric Steiner tree problem in the plane: Euclidean and rectilinear. The *Euclidean* Steiner tree problem is one of the oldest optimization problems dating back to Fermat. In this variant, distances are measured by the standard  $L_2$ -metric. In the

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<sup>†</sup>Department of Computer Science, University of Copenhagen, DK-2100 Copenhagen Ø, Denmark. E-mail: {benny,pawel,martinz}@diku.dk.

*rectilinear* version, the edges of the SMT are allowed to have horizontal and vertical directions only. This variant has important practical applications in VLSI design. We consider a generalization of the rectilinear Steiner tree problem where the edges are permitted to have any fixed number  $\lambda$  of orientations. More specifically, given  $\lambda$ , the orientations are represented by the lines making  $i\omega$  angles with the  $x$ -axis, where  $\omega = \pi/\lambda$  and  $i = 1, 2, \dots, \lambda$ . Clearly, the rectilinear Steiner tree problem is a special case of the Steiner tree problem with uniform orientations where  $\lambda = 2$ . In the last few years there has been an increasing interest in the Steiner tree problem with uniform orientations due to potential applications in VLSI design. Some very important structural properties of SMTs with uniform orientations, also referred to as  $\lambda$ -SMTs, were proved by Brazil et al. [1, 2].

Suppose that apart from the coordinates of the points that are to be interconnected, we are also told how to interconnect given points and Steiner points (their locations are unknown). We say that the *topology* of the tree is given. More specifically, we consider *full topologies* in which all terminals are leaves and all Steiner points have degree 3. It was shown by Brazil et al. [2] that for  $\lambda = 3m$ ,  $m \geq 1$ , there exists a  $\lambda$ -SMT that is a union of trees, each having a full topology (these subtrees are also called *full Steiner trees*). Clearly, each full Steiner tree is a *relatively minimal tree* for its full topology, i.e., a minimum-length tree having this topology.

In this paper we mainly address the following problem for  $\lambda = 3m$ . Given a full topology, construct a relatively minimal  $\lambda$ -tree having this topology or decide that no such tree exists. The main result of this paper is that a relatively minimal  $\lambda$ -tree exists if and only if a relatively minimal tree for the orientation-unrestricted Euclidean case exists. We prove that the locations of Steiner points in the Euclidean Steiner tree provide possible optimal locations of Steiner points in the  $\lambda$ -tree. Due to the result of Hwang [3], the existence and construction of the Euclidean relatively minimal tree for a given full topology can be performed in linear time. Consequently, the existence and construction of a relatively minimal  $\lambda$ -tree for a given full topology and  $\lambda = 3m$  can also be performed in linear time.

Another result is that for  $\lambda \neq 3m$ , the locations of Steiner points for a given topology with uniform orientations *never* coincides with the locations of the Euclidean Steiner points.

We expect that the result obtained in this paper can be incorporated into new algorithms for the Steiner tree problem with uniform orientations so that a substantial speedup for  $\lambda = 3m$  can be obtained. In fact, preliminary

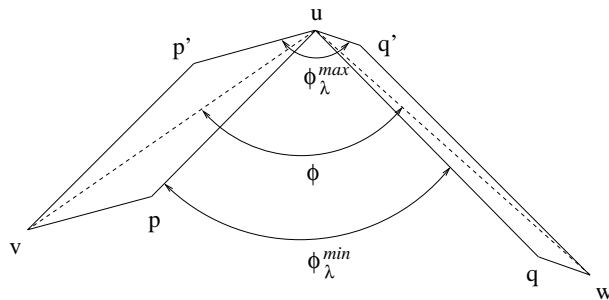


Figure 1: Two edges  $uv$  and  $uw$  incident to the same node  $u$ . The corresponding Euclidean edges are drawn as dashed line segments.

results indicate that our prototype implementation of an exact algorithm for the problem is much slower for  $\lambda = 3m$  than for other values of  $\lambda$ .

## 2 Basic definitions and results

A  $\lambda$ -edge is a shortest connection between two points  $u$  and  $v$  in which all line segments have legal directions; w.l.o.g. at most two line segments. Either one line segment in a legal direction or two line segments  $up$  and  $pv$  in legal directions; in the latter case the edge is *non-straight* (Figure 1). Similarly, an Euclidean edge is a line segment between two points, not necessarily in a legal direction.

Consider two  $\lambda$ -edges meeting at a common node  $u$ . If both edges are non-straight they can make three different angles at  $u$ , depending on how they are drawn. If  $\phi_\lambda^{\min}$  and  $\phi_\lambda^{\max}$  denote the minimum and maximum angle that can appear, the angle  $\phi$  between the Euclidean edges is clearly  $\phi_\lambda^{\min} \leq \phi \leq \phi_\lambda^{\max}$  (Figure 1).

The main results in this paper will be obtained from the following known result on  $\lambda$ -SMT angles:

**Theorem 1.** [2] *Let  $\lambda = 3m + i$ , where  $i \in \{0, 1, 2\}$ . Let  $k\omega$  ( $k$  is a positive integer) be the angle between two neighbouring  $\lambda$ -edges meeting at a Steiner point in a  $\lambda$ -SMT. Then,*

$$\begin{aligned} k \in \{2m - 1, 2m, 2m + 1\} & \text{ if } i = 0, \\ k \in \{2m, 2m + 1\} & \text{ if } i = 1, \\ k \in \{2m + 1, 2m + 2\} & \text{ if } i = 2. \end{aligned}$$

Informally, the minimum angle is the largest multiple of  $\omega$  that is strictly less than  $2\pi/3$ ; similarly, the maximum angle is the smallest multiple of  $\omega$  that is strictly larger than  $2\pi/3$ . For neighbouring edges meeting at *terminals* the lower bound still holds.

**Lemma 1.** [2] *For  $\lambda = 3m$ , let  $u$  be a node in a  $\lambda$ -SMT having two incident edges making an angle of  $(2m - 1)\omega$  with each other. Consider any of the two rays from  $u$  that makes angles  $(m - 1)\omega$  and  $m\omega$  with the edges incident at  $u$ . Then we may insert a Steiner point  $s$  on the ray, sufficiently close<sup>1</sup> to  $u$ , without changing the length of the  $\lambda$ -SMT (Figure 2a).*

This lemma implies that a Steiner point having two incident edges making an angle of  $(2m - 1)\omega$  with each other can be *moved* — without changing the length of the tree — along a direction that approximately is a bisector of this (minimum) angle. For edges meeting at a terminal this means that a Steiner point can be *created* along the same direction. Brazil et al. [2] used this so-called variational argument to prove that for  $\lambda = 3m$  there does exist a  $\lambda$ -SMT in which all angles make at least  $2\pi/3$  with each other (as in the Euclidean Steiner tree problem).

Consider two edges of a  $\lambda$ -SMT making an angle  $(2m - 1)\omega$ . The  *$\lambda$ -bisector wedge* is defined as the area given by the two rays making  $(m - 1)\omega$  and  $m\omega$  with the two edges (Figure 2a). We can now prove the following straightforward generalization of Lemma 1.

**Lemma 2.** *For  $\lambda = 3m$ , let  $u$  be a node in a  $\lambda$ -SMT having two incident edges making angle  $(2m - 1)\omega$  with each other. Then a Steiner point  $s$  may be inserted at any point in the  $\lambda$ -bisector wedge, sufficiently close to  $u$ , without changing the length of the  $\lambda$ -SMT.*

*Proof.* A Steiner point  $s$  in the  $\lambda$ -bisector wedge can be reached by performing a sequence of two Steiner point insertions along the rays defining the wedge (Figure 2b). First the Steiner point  $s'$  is inserted. The incident edges can always be made to make an angle of  $(2m - 1)\omega$  with each other — in fact to be parallel with the original edges incident at  $u$ . Then Steiner point  $s$  is inserted along the second ray.  $\square$

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<sup>1</sup>The distance that the Steiner point can be moved is always positive, but it depends on the length of the two incident line segments ( $up$  and  $uq$  in Figure 2a.)

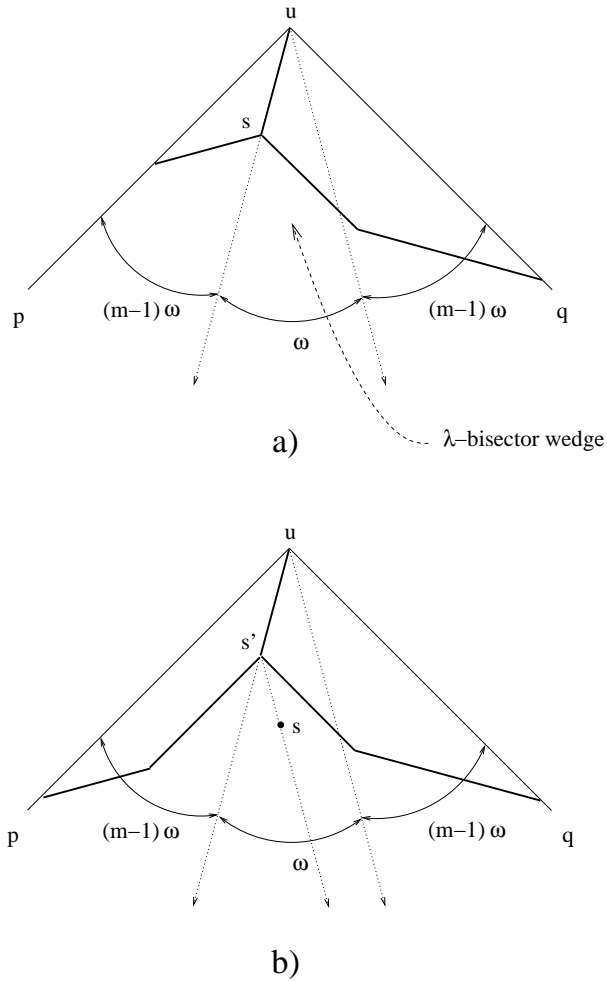


Figure 2: Inserting a Steiner point. a) Steiner point  $s$  is inserted on a ray making angles  $(m-1)\omega$  and  $m\omega$  with the edges incident at  $u$ . b) Steiner point  $s$  is inserted at a point in the  $\lambda$ -bisector wedge by a sequence of two Steiner point insertions of the former type. Note how one of the edges incident at  $s'$  is flipped in order to obtain a  $(2m-1)\omega$  angle.

### 3 Steiner points for $\lambda = 3m$

Let  $F_\lambda$  be a relatively minimal  $\lambda$ -tree for a given full topology; we assume that the full topology is *non-degenerate*, i.e., none of its relatively minimal trees has zero-length edges. Therefore, no Steiner point will overlap with a terminal or another Steiner point in such a tree. Let  $F$  be the Euclidean tree having the same topology and the same location for all terminals and Steiner points as  $F_\lambda$ , the so-called “underlying” Euclidean tree; all  $\lambda$ -edges have been straightened out in this Euclidean tree.

**Lemma 3.** *The Steiner points for  $F_\lambda$  ( $\lambda = 3m$ ) can be located such that  $F$ , the underlying Euclidean tree, has minimum length.*

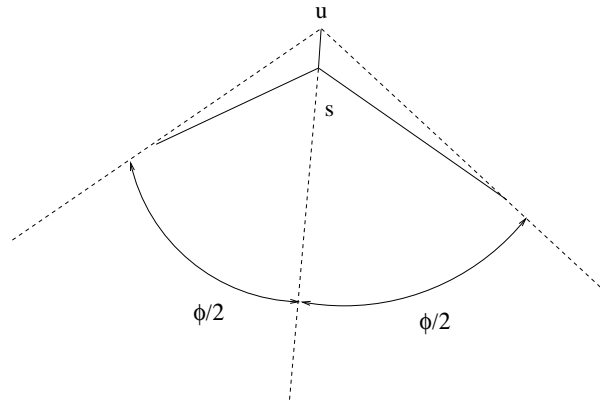
*Proof.* Assume that  $F$  is not minimal. Then there must exist a Steiner point  $u$  such that at least one pair of the incident (Euclidean) edges makes angle  $\phi < 2\pi/3$  with each other. It is well-known [4] that the Euclidean tree can be shortened by inserting a Steiner point  $s$  along the bisector of this angle (Figure 3a).

Since  $(2m - 1)\omega \leq \phi < 2\pi/3 = 2m\omega$ , the incident  $\lambda$ -edges must make  $(2m - 1)\omega$  with each other. We will now show that the (Euclidean) Steiner point  $s$  is in fact in the  $\lambda$ -bisector wedge.

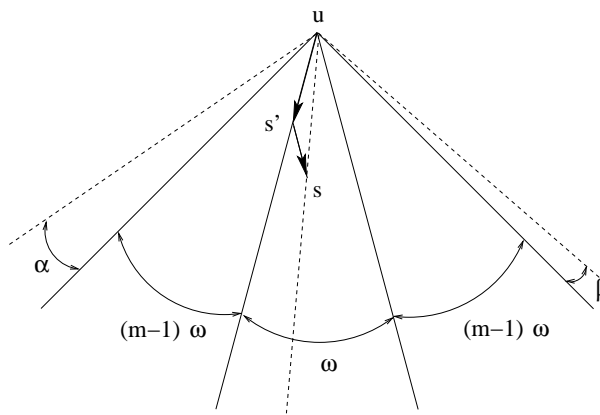
Let  $\alpha$  and  $\beta$  be the residual angles as shown in Figure 3b, such that  $\phi = (2m - 1)\omega + \alpha + \beta$ , where  $0 \leq \alpha < \omega$ ,  $0 \leq \beta < \omega$ . The bisector angle is  $\phi/2 = (m - 1/2)\omega + \alpha/2 + \beta/2$ . We have  $m\omega + \alpha > m\omega - \omega/2 + \beta/2 + \alpha \geq \phi/2$  and  $m\omega + \beta > m\omega - \omega/2 + \alpha/2 + \beta \geq \phi/2$ . Thus  $s$  is in the  $\lambda$ -bisector wedge. Lemma 2 shows that inserting  $s$  into the  $\lambda$ -tree will not change its length (assuming that  $s$  is sufficiently close to  $u$ ). We have constructed a new  $\lambda$ -tree having the same length as  $F_\lambda$  but for which the underlying Euclidean tree has strictly smaller length, a contradiction to the minimality of  $F$ .  $\square$

It follows that the Steiner points in a  $\lambda$ -tree having a given non-degenerate full topology can be assumed to be identical to the Euclidean Steiner points for the same topology. Therefore, we may use Hwang’s linear time algorithm [3] to compute the Steiner points; the terminals and Steiner points are then connected using  $\lambda$ -edges.

**Theorem 2.** *A  $\lambda$ -SMT for a non-degenerate full topology can be constructed in linear time for  $\lambda = 3m$ .*



a)



b)

Figure 3: Inserting an Euclidean Steiner point. a) Steiner point  $s$  can be inserted in order to shorten the Euclidean tree when  $\phi < 2\pi/3$ . b) Steiner point  $u$  can be moved to point  $s$  without changing the length of the  $\lambda$ -tree. (The Euclidean tree is drawn with dashed line segments.)

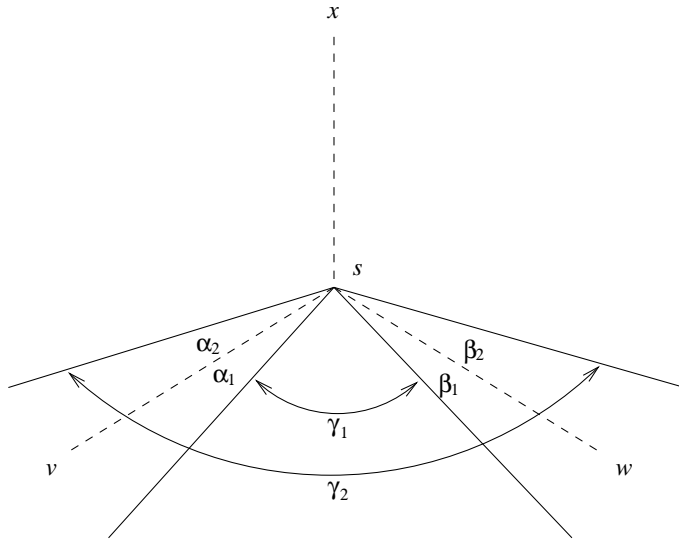


Figure 4: Lemma 4, proof illustration. The solid lines are the legal orientations from  $s$  which are closest to the direct Euclidean connections to the two points  $v$  and  $w$ . The two points  $v$  and  $w$  must be connected to  $s$  using either of these legal orientations to obtain an optimal solution.

## 4 Steiner points for $\lambda \neq 3m$

For  $\lambda \neq 3m$  the Euclidean Steiner points are *never* optimal for a  $\lambda$ -SMT in the following sense:

**Lemma 4.** *Let  $s$  be a Steiner point in a  $\lambda$ -SMT, where  $\lambda \neq 3m$ . Let the neighbours of  $s$  in the tree be  $v$ ,  $w$  and  $x$ . Then  $s$  cannot be located at the Euclidean Steiner point for  $v$ ,  $w$  and  $x$ , i.e., such that the Euclidean edges make  $2\pi/3$  with each other.*

*Proof.* Assume that the Steiner point  $s$  in the  $\lambda$ -SMT is located in the corresponding Euclidean Steiner point, such that the Euclidean edges make  $2\pi/3$  with each other. There are at least two non-straight incident edges in the  $\lambda$ -tree since otherwise there would be an angle of  $2\pi/3$  between the two straight edges and this is not possible when  $\lambda \neq 3m$ . The general situation is depicted in Figure 4. It does not matter to the following arguments whether the third edge is non-straight or not.

Now consider the angles in Figure 4. Obviously  $\alpha_1 + \alpha_2 = \omega$  and  $\beta_1 + \beta_2 =$



$\omega$  and the combined sum is simply

$$\alpha_1 + \alpha_2 + \beta_1 + \beta_2 = 2\omega = (\alpha_1 + \beta_1) + (\alpha_2 + \beta_2). \quad (1)$$

The sum  $\alpha_1 + \beta_1$  cannot be larger than  $\omega$  since that would make the angle  $\gamma_1 = 2\pi/3 - (\alpha_1 + \beta_1)$  too small according to Theorem 1. Equality would make  $2\pi/3$  a possible angle, but this contradicts  $\lambda \neq 3m$ . Consequently,  $\alpha_1 + \beta_1 < \omega$ . Similarly it can be shown that  $\alpha_2 + \beta_2 < \omega$  using Theorem 1 on the angle  $\gamma_2$ . Now we have that  $(\alpha_1 + \beta_1) + (\alpha_2 + \beta_2) < 2\omega$  which clearly contradicts (1). Therefore  $s$  cannot be a Steiner point in the  $\lambda$ -SMT.  $\square$

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