Technical Report DIKU-TR-98-24
Department of Computer Science, University of Copenhagen
Universitetsparken 1, DK-2100 Copenhagen East, Denmark
http://www.diku.dk

Classification of toppoints for the gradient squared

Ole Fogh Olsen* Peter Johansen†

Abstract

In this paper we present a classification of the toppoints for the gradient squared. In this approach the parameters for the critical curve are derived from implicit differentiation of the defining equations. The two generic events for the gradient squared in a family of images are: (1) the creation or annihilation of a saddle and an extremum, (2) two minima and a saddle merge or they split into one minimum.

1 Introduction

We study the evolution of critical points for some functional of the image L. We are interested in topological change of the critical point configuration. The evolution arises when the image evolves according to some partial differential equation with the raw image I as boundary condition. The heat equation is a partial differential equation which has our special interest for image evolution:

$$L_t(x, y; t) = L_{xx}(x, y; t) + L_{yy}(x, y; t), \qquad L(x, y; 0) = I$$

*fogh@diku.dk, http://www.diku.dk/users/fogh/
†peterjo@diku.dk, http://www.diku.dk/users/peterjo/

Partial derivatives are denoted with lower-case indices. The solution is called a scale-space image. For a general evolution we have a family of images $L(x, y; t)$ indexed by t and with domain variables x, y as a solution to a PDE. The critical points for the image are defined by the equations:

$$L_x(x, y, t) = 0 \wedge L_y(x, y, t) = 0 \tag{1}$$

In the (x, y, t)-space the two equations implicitly define a curve. We denote it the critical curve.

The used scheme calculates the derivatives of the critical curve $r(s) = (x(s), y(s), t(s))$ from the derivative of the family of images by implicit differentiation. Hence the above equations are derived with respect to the curve parameter s:

$$\begin{aligned} \frac{d^n}{ds^n} L_x(x(s), y(s), t(s)) &= 0 \\ \frac{d^n}{ds^n} L_y(x(s), y(s), t(s)) &= 0 \end{aligned}$$

We want to study the local structure of these solutions in the neighbourhood of toppoints. A toppoint is a point on the critical curve where the determinant of the Hessian is zero:

$$L_{xx}(r)L_{yy}(r) - L_{xy}^2(r) = 0$$

The above equation and the defining equations (1) determine the toppoint. A toppoint marks a position in (x, y, t) space where the topology of the image changes. At a toppoint critical points appear or disappear as a function of t. Johansen [1] has classified the toppoints of the image both for general evolution and in the specific case of scale-space images according to the type of the intersection of the branches of the critical curve.

In this paper we want to study the local structure at toppoints for the gradient squared g when the image evolve. Hence, the following functional is under investigation :

$$g(x, y, t) = L_x(x, y, t)^2 + L_y(x, y, t)^2 \tag{2}$$

Thus the above equations will be applied to g. The generic results for the gradient squared have been obtained by Olsen et al. [2]. That is whether a certain type of topology change can occur generically. The emphasis in that paper is on scale-space images. It is shown that the generic topology changes "can go both ways" meaning that critical points can appear or disappear generically. The result on appearing and disappearing critical point is reached using heat polynomials as the local model.

The knowledge of the generic topological changes has been utilised by Olsen et al. [3] for multi-scale segmentation. Each local minimum in the gradient squared is assigned a segment. The change of segment topology with scale is derived from the change of minima topology. Based on this the segments are linked over scale. The linking makes it possible to segment the coarse scale structures and localise the segments on a fine scale.

Pratikakis et al. [4] has also based their segmentation scheme on the knowledge of the generic changes.

2 Gradient Squared Evolution

We want to study the local structure at toppoints for the gradient squared denoted g, see equation 2. In particularly we are interested in the case where the image L evolves according to the heat equation but the derived results will hold for a general evolution. We evaluate all expressions in a Cartesian coordinate system with origin at the toppoint and with the (x, y) directions oriented according to the local geometry in order to simplify expressions.

On the critical curves the spatial derivatives of g equals 0 :

$$\begin{pmatrix} g_x \\ g_y \end{pmatrix} = 2 \begin{pmatrix} L_{xx} & L_{xy} \\ L_{xy} & L_{yy} \end{pmatrix} \begin{pmatrix} L_x \\ L_y \end{pmatrix} = 0 \tag{3}$$

From Equations 3 it is clear we can split the solutions in two cases, namely case A in which the gradient for L equals zero (critical points for the image) and in case B in which the Hessian for L degenerates in the direction of the gradient of L (perpendicular intersection of flow lines and parabolic lines of the image).

For later use we write derivatives using indices and implicit summation over i for the variables x and y:

$$\begin{aligned} g_j &= 2L_i L_{ij} \\ g_{jk} &= 2L_{ik}L_{ij} + 2L_i L_{ijk} \\ g_{jkl} &= 2L_{ikl}L_{ij} + 2L_{ik}L_{ijl} + 2L_{il}L_{ijk} + 2L_i L_{ijkl} \\ g_{jklm} &= 2L_{iklm}L_{ij} + 2L_{ikl}L_{ijm} + 2L_{ikm}L_{ijl} + 2L_{ik}L_{ijlm} \end{aligned} \tag{4}$$

$$+ \quad 2L_{ilm}L_{ijk} + 2L_{il}L_{ijkm} + 2L_{im}L_{ijkl} + L_iL_{ijklm}$$

We assume the critical curve $r(s) = (x(s), y(s), t(s))$ is parametrised with a parameter s. Now we want to study the tangent of the critical curves. If we substitute g for L in the defining equations 1 and differentiate with respect to s we have:

$$\frac{dg_x}{ds} = x'g_{xx} + y'g_{xy} + t'g_{xt} = 0 \quad (5)$$

$$\frac{dg_y}{ds} = x'g_{xy} + y'g_{yy} + t'g_{yt} = 0 \quad (6)$$

Where $'$ indicates derivation with respect to s. We can continue differentiating to gain information on the higher order derivatives of the curve:

$$\frac{d^2g_x}{ds^2} = x'\nabla g_{xx}r' + y'\nabla g_{xy}r' + t'\nabla g_{xt}r' + x''g_{xx} + y''g_{xy} + t''g_{xt} = 0 \quad (7)$$

$$\frac{d^2g_y}{ds^2} = x'\nabla g_{xy}r' + y'\nabla g_{yy}r' + t'\nabla g_{yt}r' + x''g_{xy} + y''g_{yy} + t''g_{yt} = 0 \quad (8)$$

We will also normalise the parametrisation such the curve is parametrised by arc length:

$$x'^2 + y'^2 + t'^2 = 1$$

2.1 Case A

In case A, being on a critical curve for the gradient squared g implies that $L_x = 0 \wedge L_y = 0$. From equation 4 we have the elements of the Hessian and now we evaluate for case A.

$$\begin{aligned} g_{xx} &= 2(L_{xx}^2 + L_{xy}^2) \\ g_{xy} &= 2(L_{xx} + L_{yy})L_{xy} \\ g_{yy} &= 2(L_{xy}^2 + L_{yy}^2) \end{aligned} \quad (9)$$

By selecting orthogonal (x, y) directions at a toppoint where the Hessian for g has rank 1 we can have $g_{xx} \neq 0 \wedge g_{xy} = 0 \wedge g_{yy} = 0$ which implies by equations 9 that $L_{xy} = 0 \wedge L_{yy} = 0 \wedge L_{xx} \neq 0$. Hence, the chosen directions for (x, y) are the principal curvature directions for both the image L and the gradient squared g.

Next we calculate the remaining terms in the equations 5 and 6 expressed in L-derivatives:

$$\begin{aligned} g_{xt} &= 2L_{xt}L_{xx} \\ g_{yt} &= 2(L_{xt}L_{xy} + L_{yt}L_{yy} + L_xL_{xyt} + L_yL_{yyt}) = 0 \end{aligned}$$

Equation 6 is vacuous and does not reveal anything about r'. We need another equation. We evaluate equations 7 and 8 under the case constraints:

$$\begin{aligned} \frac{d^2g_x}{ds^2} &= x'\nabla g_{xx}r' + y'\nabla g_{xy}r' + t'\nabla g_{xt}r' + x''g_{xx} + t''g_{xt} = 0 \\ \frac{d^2g_y}{ds^2} &= x'\nabla g_{xy}r' + y'\nabla g_{yy}r' + t'\nabla g_{yt}r' = 0 \end{aligned} \tag{10}$$

Equation 10 seems promising for determine r'. So the involved g-derivatives are calculated in terms of L-derivatives using the constraints $L_x = L_y = L_{xy} = L_{yy} = 0$:

$$\begin{aligned} g_{xxx} &= 6L_{xxx}L_{xx} \\ g_{xxy} &= 4L_{xxy}L_{xx} \\ g_{xyy} &= 2L_{xyy}L_{xx} \\ g_{yyy} &= 0 \\ g_{xyt} &= 2(L_{xyt}L_{xx} + L_{xt}L_{xxy} + L_{yt}L_{xyy}) \\ g_{yyt} &= 2(L_{xt}L_{xyy} + L_{yt}L_{yyy}) \\ g_{ytt} &= 4(L_{xt}L_{xyt} + L_{yt}L_{yyt}) \end{aligned}$$

Thus g_{yyy} is always zero at toppoint for this type of critical curve! This reduces equation 10 to:

$$\begin{aligned} \frac{d^2}{ds^2}g_y &= g_{xxy}x'^2 + 2g_{xyy}x'y' + 2g_{xyt}x't' + 2g_{yyt}y't' + g_{ytt}t'^2 \\ &= 4(x'^2L_{xx}L_{xxy} + x'y'L_{xx}L_{xyy} \\ &+ x't'(L_{xt}L_{xxy} + L_{xx}L_{xyt} + L_{xyy}L_{yt}) \\ &+ y't'(L_{xt}L_{xyy} + L_{yt}L_{yyy}) + t'^2(L_{xt}L_{xyt} + L_{yt}L_{yyt})) = 0 \end{aligned} \tag{11}$$

and equation 5 was in term of L :

$$2L_{xx}^2 x' + 2L_{xt}L_{xx}t' = 0 \Rightarrow x' = -t'\frac{L_{xt}}{L_{xx}} \tag{12}$$

Substituting equation 12 into equation 11 and assuming $L_{yyy} \neq 0$ we get two solutions:

$$r' = (0, y', 0)$$

$$r' = (-t'\frac{L_{xt}}{L_{xx}}, t'\frac{L_{xt}L_{xyy} - L_{xx}L_{yyt}}{L_{xx}L_{yyy}}, t')$$

Hence there are two tangents through the toppoint, one in the y-direction and one in a transverse direction.

We can use a normalisation of the curve parameter for fixing the last degree of freedom in r'. We get two solutions with same absolute value but opposite signs. They correspond to opposite traversals of the curve. We just show one here. First solution is :

$$r' = (0, 1, 0)$$

The second solution is bit more complex.
Let $b = L_{xt}L_{xyy} - L_{xx}L_{yyt}$, $a = (L_{xt}^2 L_{yyy}^2 + b^2 + L_{xx}^2 L_{yyy}^2)^{\frac{1}{2}}$ and solve $r'^2 = 1$:

$$t' = \left(\frac{L_{xx}^2 L_{yyy}^2}{L_{xt}^2 L_{yyy}^2 + b^2 + L_{xx}^2 L_{yyy}^2}\right)^{1/2}$$

$$r' = \frac{1}{a}\begin{pmatrix} -L_{yyy}\ L_{xt} \\ b \\ L_{xx}L_{yyy} \end{pmatrix} \tag{13}$$

2.1.1 Classifying the type of points on the curves

Next we want to classify the type of critical point on the curves through the toppoint. We follow Johansen and approximate the determinant of the Hessian. We have two cases corresponding to the two different tangent directions (one solely in the y direction and one with components in all directions including t) .

$$Det(H_g) = \begin{vmatrix} g_{xx} + \nabla g_{xx} r's & \nabla g_{xy} r's \\ \nabla g_{xy} r's & \nabla g_{yy} r's + \frac{1}{2}(\nabla^2 g_{yy} r'^{\otimes 2} + \nabla g_{yy} r'')s^2 \end{vmatrix} + O(s^3) \tag{14}$$

First solution

For the first solution $r' = (0,1,0)$ equation 14 evaluates to:

$$\begin{aligned} Det(H_g) &= \begin{vmatrix} g_{xx} + g_{xxy}s & g_{xyy}s \\ g_{xyy}s & \frac{1}{2}(g_{yyyy} + g_{xyy}x'' + g_{yyt}t'')s^2 \end{vmatrix} + O(s^3) \\ &= s^2(\frac{1}{2}g_{xx}(g_{yyyy} + g_{xyy}x'' + g_{yyt}t'') - g_{xyy}^2) + O(s^3) \end{aligned} \quad (15)$$

One more equation is derived by implicit differentiation of equation 8 and simplified according to case conditions and the specific tangent solution:

$$\begin{aligned} \frac{d^3g_y}{ds^3} &= \nabla^3 g_y r' \otimes r' \otimes r' + 3\nabla^2 g_y r' \otimes r'' + \nabla g_y r''' \\ &= g_{yyyy} + 3g_{yyx}x'' + 3g_{yyt}t'' = 0 \quad (16) \\ &\Rightarrow \\ -g_{yyyy}/3 &= g_{yyx}x'' + g_{yyt}t'' \end{aligned}$$

We substitute the result into equation 15:

$$\begin{aligned} Det(H_g) &= s^2(\frac{1}{2}g_{xx}(g_{yyyy}(1-\frac{1}{3}) - g_{xyy}^2) + O(s^3) \\ &= 4s^2L_{xx}^2L_{yyy}^2 + O(s^3) \end{aligned}$$

The determinant is always positive, consequently we conclude that locally the curve consists of only extremum points. The type of extremum is determined by the sign of the trace of the Hessian:

$$Trace(H_g) = g_{xx} + O(s) = 2L_{xx}^2 + O(s)$$

Since locally the trace is always positive we conclude that the curve consists exclusively of minimum points.

Second solution

For the second solution given in equation 13 we have:

$$\begin{aligned} det(H_g) &= s\ g_{xx}(-g_{xyy}L_{yyy}L_{xt} + g_{yyt}L_{xx}L_{yyy})/a + O(s^2) \\ &= s\ 4L_{xx}^3L_{yyy}^2L_{yt}/a + O(s^2) \\ a &= (L_{xt}^2L_{yyy}^2 + (L_{xt}L_{xyy} - L_{xx}L_{yyt})^2 + L_{xx}^2L_{yyy}^2)^{\frac{1}{2}} \end{aligned}$$

We assume $L_{yt} \neq 0$. Consequently the determinant is only zero for $s = 0$. The sign of the determinant changes when the s-parameter passes zero. Hence the curve consists of saddle points and extremum points and the switch between the two types happens for $s = 0$.

The trace of the Hessian is :

$$Trace(H_g) = g_{xx} + O(s) = 2L_{xx}^2 + O(s)$$

Hence the curve consist of saddle points and minimum points.

2.1.2 The t-direction

The t-axis has special meaning since it describes an evolution. It is meaningful to give this axis a direction (in contrast to the spatial x, y-axis). We want to know if critical points can both appear and disappear when the parameter t increases. In the image evolution this corresponds to appearing and disappearing structure.

First solution

For the first solution we calculated that $t(s) = O(s^2)$. We determine t'' to gain some information on the behaviour in the t-direction. We use equation 7 which reduces for this solution to:

$$\frac{d^2}{ds^2} g_x = g_{xyy} + g_{xx}x'' + g_{xt}t'' = 0 \tag{17}$$

Solving equations 17 and 16 for x'' and t'' and using $g_{yyyy} = 6(L_{xyy}^2 + L_{yyy}^2)$ we have

$$\begin{aligned} x'' &= \frac{-L_{xyy}L_{yt} + L_{xt}L_{yyy}}{L_{xx}L_{yt}} \\ t'' &= \frac{-L_{yyy}}{L_{yt}} \end{aligned}$$

When $t'' > 0$ the curve is concave, and if $t'' < 0$ the curve is convex.

Second solution

We have calculated in equation 13 that $t(s) = sL_{xx}L_{yyy}/a + O(s^2)$. Since the sign of $t(s)$ changes with s, both positive and negative t values occur on the same curve.

2.1.3 Both solutions

Now we want to relate the two curves (the two solutions) which of course are related through the image derivatives. We want to determine the possible configurations of the two curves. The t-component of the two curves are:

$$
\begin{aligned}
t_1(s) &= \frac{-L_{yyy}}{L_{yt}} s^2 + O(s^3) \\
t_2(s) &= s L_{xx} L_{yyy}/a + O(s^2)
\end{aligned}
$$

First we assume that $\frac{-L_{yyy}}{L_{yt}} > 0$. Hence the first curve is concave. Next we check the positive t-part of the second curve. That is, we assume $sL_{xx}L_{yyy}/a > 0$. Since a is positive it implies $sL_{xx}L_{yyy} > 0$. The determinant of the Hessian evaluates to:

$$
sign(det(H_g)) = sign(sL_{xx}^3 L_{yyy}^2 L_{yt}) = sign(sL_{xx}L_{yyy}) sign(L_{yyy}L_{yt}) = negative
$$

Hence the positive part of the second curve consists of saddle points when the first curve is concave. The negative part of the second curve consists of minimum points when the first curve is concave.

In the opposite case $\frac{-L_{yyy}}{L_{yt}} < 0$ the reasoning follows the same line. The first curve is convex. The positive part of the second curve consists of minimum points and the negative part consists of saddle points.

So we always have two minima (the first curve) meeting a saddle (second curve) and becoming a minimum point (second curve). This event can occur either as an annihilation (three critical points merge into one) or as a creation (one become three). See figure 1 for an example.

2.1.4 Summary

Case A occurs in a critical point for L. It is assumed that the Hessian for L has rank equal one. For convenience the coordinate axis are chosen to be the principal curvature directions and such that the curvature in x-direction is non-vanishing. The conditions were the following:

$$
\begin{aligned}
L_x &= 0 \ \wedge \ L_y = 0 \ \wedge \ L_{xy} = 0 \ \wedge \ L_{yy} = 0 \\
L_{xx} &\neq 0 \ \wedge \ L_{yyy} \neq 0 \ \wedge \ L_{yt} \neq 0
\end{aligned}
$$

In the case of Gaussian scale space we have $L_{yt} = L_{xxy} + L_{yyy} \neq 0$. The case conditions lead to two intersecting tangents. One along the y-axis and one

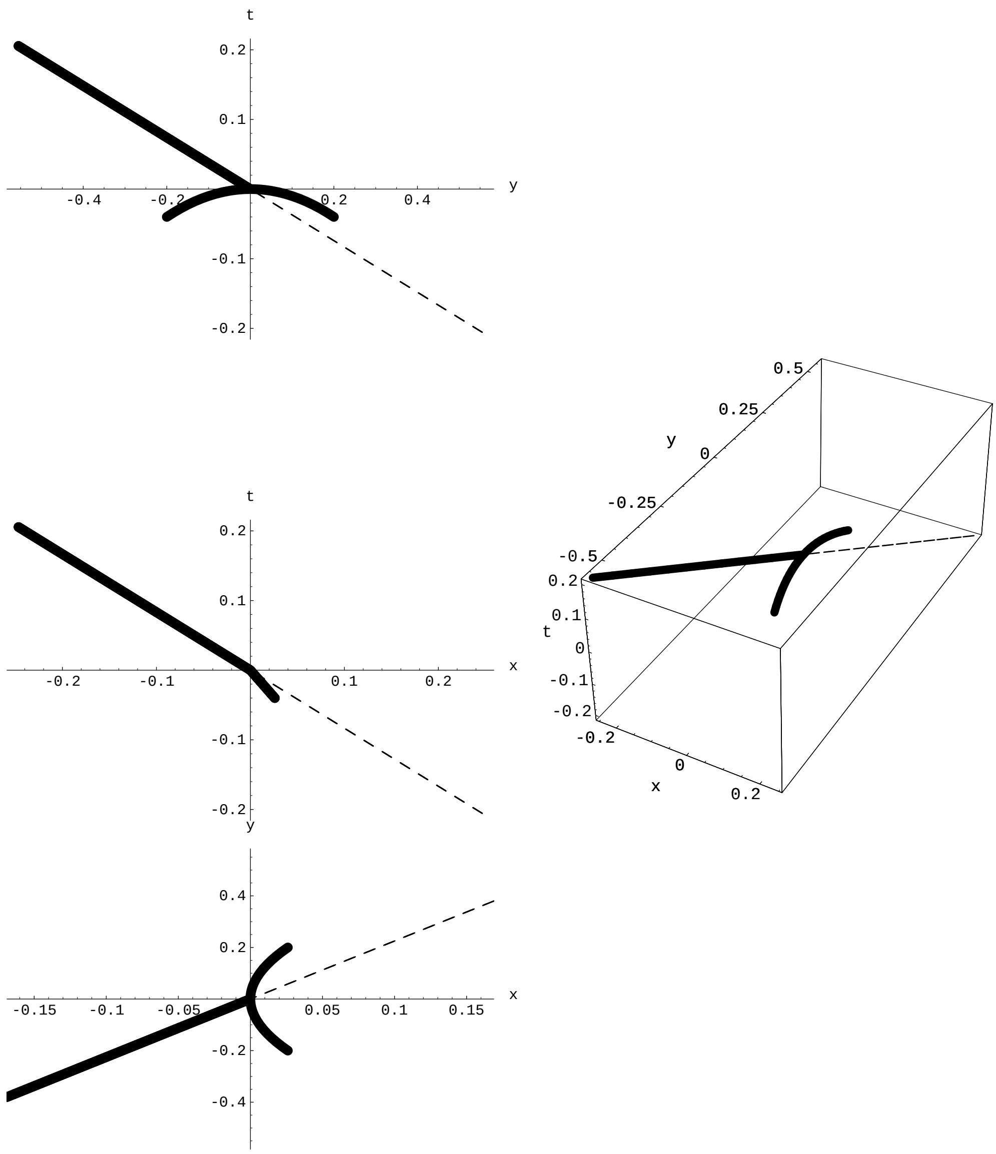

Figure 1: Two minima and a saddle merge into one minimum for increasing t. The saddle points are plotted with a dotted line and the minima points with full lines. To the left is depicted the projection to the three axes-planes (y,t), (x,t) and (x,y). In the right column is plotted the curves in "3D". The coefficients for this plot are: $L_{xx} = 1, L_{xyy} = 1.1, L_{xt} = 1.2, L_{yyt} = 1.05, L_{yyy} = -0.1, L_{yt} = -0.05$. This example shows a merge of critical points as predicted by the algebra since $-L_{yyy}/L_{yt}$ is negative.

not contained in any of the axis-planes. Second order derivatives for the first solution reveal the behaviour in the t-direction. There are two configurations: Two minima (the first curve) meet a saddle (the second curve) and change into a single minimum point (second curve) when t increases in value. An example of this event is plotted in figure 1. The reverse event is also possible: one minimum point changes into two minima and a saddle with increasing t-values. Simply reverse the t-direction in figure 1 and you have the latter event.

2.2 Case B

In this case the Hessian for image L degenerates in the direction of the gradient for the image L. By a rotation of the coordinate system we choose the gradient direction to be the y direction. In this coordinate system the constraints are:

$$L_y \neq 0 \wedge L_x = 0 \wedge L_{xy} = 0 \wedge L_{yy} = 0 \wedge L_{xx} \neq 0$$

The above constraints and equation 3 imply that the gradient of g equals zero. We calculate the needed g-derivatives under these conditions.

$$\begin{aligned}
g_{xx}/2 &= L_{xx}L_{xx} + L_y L_{xxy} \\
g_{xy}/2 &= L_y L_{xyy} \\
g_{yy}/2 &= L_y L_{yyy} \\
g_{xt}/2 &= L_{xt}L_{xx} + L_y L_{xyt} \\
g_{yt}/2 &= L_y L_{yyt} \\
g_{xyy} &= 2L_{xyy}L_{xx} + 2L_y L_{xyyy} \\
g_{yyy} &= 2L_y L_{yyyy}
\end{aligned}$$

The determinant of the Hessian of g is zero at the toppoints

$$g_{xx}g_{yy} - g_{xy}^2 = (L_{xx}L_{xx} + L_y L_{xxy})L_y L_{yyy} - L_y^2 L_{xyy}^2 = 0$$

By implicit differentiation once we have:

$$\begin{aligned}
g_{xx}x' + g_{xy}y' + g_{xt}t' &= 0 \\
g_{xy}x' + g_{yy}y' + g_{yt}t' &= 0
\end{aligned}$$

We assume $g_{xx}g_{yt} - g_{xt}g_{xy} \neq 0$. This is generic since $g_{xx}g_{yt} - g_{xt}g_{xy} = 4(L_{xx}L_{xx} + L_yL_{xxy})(L_yL_{yyt}) - 4(L_{xt}L_{xx} + L_yL_{xyt})(L_yL_{xyy})$ is not in general zero.

$$t' = y'\frac{g_{xx}g_{yy} - g_{xy}^2}{g_{xt}g_{xy} - g_{xx}g_{yt}} = 0$$

Let us assume $L_{yyy} \neq 0$ then

$$y' = -x'\frac{L_{xyy}}{L_{yyy}}$$

Normalisation of the tangent now gives:

$$\begin{aligned} x'^2 + y'^2 &= x'^2(1 + \frac{L_{xyy}^2}{L_{yyy}^2}) = 1 \Rightarrow \\ x' &= \frac{L_{yyy}}{\sqrt{L_{xyy}^2 + L_{yyy}^2}} \\ y' &= \frac{-L_{xyy}}{\sqrt{L_{xyy}^2 + L_{yyy}^2}} \end{aligned}$$

The tangent to the critical curve lies in the x-y-plane and the direction relative to the L gradient direction y is given by the above formula. After normalising the tangent we have two solutions. Each solution corresponds to a direction of traversal. So essentially we have one tangent direction. Next we want to determine the type of points (extremum or saddle) on the curve:

$$\begin{aligned} Det(H_g)(s) &= \begin{vmatrix} g_{xx} & 0 \\ 0 & \nabla g_{yy} r' s \end{vmatrix} + O(s^2) \\ &= g_{xx}(g_{xyy}\frac{L_{yyy}}{\sqrt{L_{xyy}^2 + L_{yyy}^2}} + g_{yyy}\frac{-L_{xyy}}{\sqrt{L_{xyy}^2 + L_{yyy}^2}})s + O(s^2) \\ &= s \;\; 2(L_{xx}^2 + L_yL_{xxy}) \\ &\quad \frac{(2L_{xyy}L_{xx} + 2L_yL_{xyyy})L_{yyy} - 2L_yL_{yyyy}L_{xyy}}{\sqrt{L_{xyy}^2 + L_{yyy}^2}} + O(s^2) \end{aligned}$$

We conclude that the type of the critical point changes with sign of s. Hence the curve consists of saddle points and extremum points. A new set of equations can be set up by implicit differentiation once more. By this t'' can be calculated and it can be determined that both positive and negative values are generic. This means both creation and annihilation events can occur.

References

[1] Peter Johansen. On the classification of toppoints in scale space. *Journal of Mathematical Imaging and Vision*, 4:57–67, 1994.

[2] Ole Fogh Olsen and Mads Nielsen. Generic events for the gradient squared with application to multi-scale segmentation. In *Scale-Space Theory in Computer Vision, Proc. 1st International Conference*, volume 1252 of *Lecture Notes in Computer Science*, pages 101–112, Utrecht, The Netherlands, July 1997.

[3] Ole Fogh Olsen and Mads Nielsen. Multi-scale gradient magnitude watershed segmentation. In *ICIAP'97 - 9th International Conference on Image Analysis and Processing*, Lecture Notes in Computer Science, Florence, Italy, September 1997.

[4] I. E. Pratikakis, H. Saleli, and J. Cornelis. Hierarchical segmentation using dynamics of multiscale gradient watersheds. 1998. Submitted.