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Abstract

We consider ray bundles emanating from a source such as a camera or light source. We derive the full spatial and temporal structure to first order of the intersection of ray bundles with scene geometry, where scene geometry given as any implicit function. Further, we present the full details of 2 often used geometrical representations. The first order structure may be used as the linear approximation of the change of photons as the camera, objects, and light source change as function of space and time. Our work generalises previous work on ray differentials [Igehy, 1999] and photon differentials [Schjøth et al., 2007].

1 Ray differential

In this work we consider reflection and refraction of light rays off and through surfaces as illustrated in Figure 1. We will derive the spatial-temporal first order structure of these processes without any simplifying assumptions. Consider a point on a ray and its direction $\boldsymbol{P}, \boldsymbol{V} \in \mathbb{R}^3$, and a simple 2 dimensional surface such as a plane or a sphere embedded in 3-space, $\boldsymbol{x} \in \mathbb{R}^3$, such that

$$\boldsymbol{v} = \boldsymbol{x} - \boldsymbol{P}, \tag{1a}$$

$$\boldsymbol{V} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}. \tag{1b}$$

We use column vectors, hence $\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v}^T\boldsymbol{v}}$. Following [Igehy, 1999, Schjøth et al., 2007] we calculate the partial derivative of $\boldsymbol{P}$ and $\boldsymbol{V}$ w.r.t. $\boldsymbol{x}$, and we will use the notation of differentials [Magnus and Neudecker, 1988]. Differentials are rooted in Taylor series, i.e. consider an analytical function $f : \mathbb{R} \to \mathbb{R}$, and write its Taylor series as, $f(x + \Delta x) = f(x) + f'(x)\Delta x + \mathcal{O}(\Delta x^2)$, where $\mathcal{O}$ is the remainder in Landau notation, and f' is the first order derivative of f. We may reorder, assume infinitesimal small values of Δx's and ignore the even smaller remainder and define,

$$df = f'(x)dx. \tag{2}$$

The extension to vector and matrix equations is straight forward, since their Taylor series are element wise Taylor series. We use the same notation except the derivative now is the Jacobian matrix, e.g. for vector equations such as $\boldsymbol{V} \in \mathbb{R}^n \to \mathbb{R}^m$ and $\boldsymbol{x} \in \mathbb{R}^n$, the Jacobian of $\boldsymbol{V}$ w.r.t. the variable $\boldsymbol{x}$ is $D_{\boldsymbol{x}}\boldsymbol{V}$ who's ij'th entry is $\frac{\partial \boldsymbol{V}_i}{\partial \boldsymbol{x}_j}$. Hence, the j'th column is the change vector of V when only considering the j'th coordinate direction. The Jacobian w.r.t. the full space of parameters is often just written as $D\boldsymbol{V}$ for convenience. The differential embodies the full first order structure of a function, and a first order estimate of the change is obtained by replacing the infinitesimals with finite values, i.e. dx with Δx.

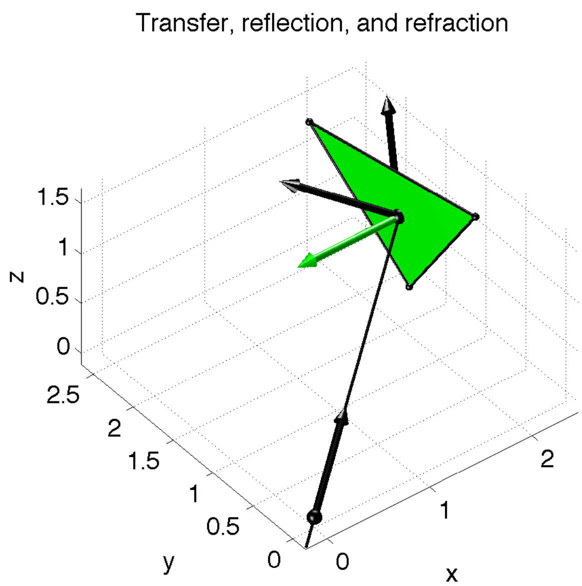

Figure 1: Transfer, Reflection, and Refraction by a flat surface. Black arrows are ray directions from origin, through reflection and refraction, green arrow is the normal for the green surface patch.

From (1) we may calculate the differential of $\boldsymbol{V}$ as,

$$d\boldsymbol{V} = \frac{(d\boldsymbol{v})(\boldsymbol{v}^T\boldsymbol{v})^{1/2} - \boldsymbol{v}(\boldsymbol{v}^T\boldsymbol{v})^{-1/2}\boldsymbol{v}^T d\boldsymbol{v}}{\boldsymbol{v}^T\boldsymbol{v}} \tag{3a}$$

$$= \frac{\boldsymbol{v}^T\boldsymbol{v}\boldsymbol{I}_3 - \boldsymbol{v}\boldsymbol{v}^T}{(\boldsymbol{v}^T\boldsymbol{v})^{3/2}}\, d\boldsymbol{v} \tag{3b}$$

$$= \frac{\boldsymbol{v}^T\boldsymbol{v}\boldsymbol{I}_3 - \boldsymbol{v}\boldsymbol{v}^T}{(\boldsymbol{v}^T\boldsymbol{v})^{3/2}}\, (d\boldsymbol{x} - d\boldsymbol{P}), \tag{3c}$$

where $\boldsymbol{I}_3$ is the 3×3 identity matrix. We are now able to calculate the complete first order structure of $\boldsymbol{V}$ given the first order structure of $\boldsymbol{P}$ and $\boldsymbol{x}$, and with that, we are able to make linear approximation of the changes in $\boldsymbol{V}$ given the changes in $\boldsymbol{P}$ and $\boldsymbol{x}$. E.g. if $\boldsymbol{P}$ is constant, then $d\boldsymbol{P} = 0$, and

$$d\boldsymbol{V} = \frac{\boldsymbol{v}^T\boldsymbol{v}\boldsymbol{I}_3 - \boldsymbol{v}\boldsymbol{v}^T}{(\boldsymbol{v}^T\boldsymbol{v})^{3/2}}\, d\boldsymbol{x}, \tag{4}$$

From this form, we can easily identify the matrix of partial derivatives as

$$\frac{d\boldsymbol{V}}{d\boldsymbol{x}} = \frac{\boldsymbol{v}^T\boldsymbol{v}\boldsymbol{I}_3 - \boldsymbol{v}\boldsymbol{v}^T}{(\boldsymbol{v}^T\boldsymbol{v})^{3/2}}. \tag{5}$$

The partial derivative of $\boldsymbol{V}$ w.r.t. $\boldsymbol{P}$ is similarly found as $-\frac{d\boldsymbol{V}}{d\boldsymbol{x}}$.

If $\boldsymbol{x}$ is a plane, then a natural parametrization will be a set of orthogonal axes spanning the plane, and to continue the example assume that the plane is orthogonal to the third axis, then

$$d\boldsymbol{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} \tag{6}$$

Assuming that we are currently viewing in direction $\boldsymbol{v}^*$ which passes through coordinate $\begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ and corresponding to $\boldsymbol{V}^*$, then the first order approximation to $\boldsymbol{V}^* + \Delta\boldsymbol{V}$ when $\begin{bmatrix} x_1 + \Delta x_1 & x_2 + \Delta x_2 \end{bmatrix}^T$ is calculated by evaluating $\Delta\boldsymbol{V}$ using $dx_1 = \Delta x_1$ and $dx_2 = \Delta x_2$, i.e.

$$\Delta\boldsymbol{V} = \frac{\boldsymbol{v}^T\boldsymbol{v}\boldsymbol{I}_3 - \boldsymbol{v}\boldsymbol{v}^T}{(\boldsymbol{v}^T\boldsymbol{v})^{3/2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}. \tag{7}$$

If $\boldsymbol{x}$ instead is a sphere of radius 1, we may more naturally use the spherical parametrization,

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{bmatrix}, \tag{8}$$

such that

$$\begin{aligned} d\boldsymbol{x} &= \begin{bmatrix} -\sin\phi\sin\theta\,d\phi + \cos\phi\cos\theta\,d\theta \\ \cos\phi\sin\theta\,d\phi + \sin\phi\cos\theta\,d\theta \\ -\sin\theta\,d\theta \end{bmatrix} && \text{(9a)} \\ &= \begin{bmatrix} -\sin\phi\sin\theta & \cos\phi\cos\theta \\ \cos\phi\sin\theta & \sin\phi\cos\theta \\ 0 & -\sin\theta \end{bmatrix} \begin{bmatrix} d\phi \\ d\theta \end{bmatrix} && \text{(9b)} \\ &= \boldsymbol{A}\,d\boldsymbol{\theta}, && \text{(9c)} \end{aligned}$$

where $\boldsymbol{A}$ and $\boldsymbol{\theta}$ are defined as indicated above. As a note, parallel rays can be implemented by enforcing $d\boldsymbol{V} = 0$.

The differentials are in no way limited to static scenes and cameras. In (6) we may add a time derivative as

$$d\boldsymbol{x} = \begin{bmatrix} D_{x_1}\boldsymbol{x} & D_{x_2}\boldsymbol{x} & D_t\boldsymbol{x} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dt \end{bmatrix}, \tag{10}$$

where $D_i\boldsymbol{x}$ are vectors of partial derivatives as indicated. I.e. if the plane moves with a unit speed along the third coordinate axis, then $D_t\boldsymbol{x} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$, and if we wish to estimate the change in $\boldsymbol{V}$ as after 1 unit of time, then we evaluate using $dt = 1$.

2 Transfer, Reflection, and Refraction

In the following we will investigate light's interaction with dielectric material, i.e. reflection and refraction. We consider a ray from a source at point $\boldsymbol{P}$ with direction $\boldsymbol{V}$, which intersection a surface at position $\boldsymbol{Q}$ and is reflection and refraction in directions $\boldsymbol{W}_{\text{reflect}}$ and $\boldsymbol{W}_{\text{refract}}$ respectively. The normal at $\boldsymbol{Q}$ will be denoted $\boldsymbol{N}$, and the refraction ration at $\boldsymbol{Q}$ will be denoted η. Our main goal will be to calculate the differentials, $d\boldsymbol{Q}$, $d\boldsymbol{W}_{\text{reflect}}$, and $d\boldsymbol{W}_{\text{refract}}$ as a function of relevant parametrizations e.g.

$$\begin{aligned} d\boldsymbol{Q} &= D_{\boldsymbol{V}}\boldsymbol{Q}\,d\boldsymbol{V} + D_{\boldsymbol{P}}\boldsymbol{Q}\,d\boldsymbol{P} + D_{\boldsymbol{N}}\boldsymbol{Q}\,d\boldsymbol{N} + D_\eta\boldsymbol{Q}\,d\eta && \text{(11a)} \\ d\boldsymbol{W}_{\text{reflect}} &= D_{\boldsymbol{V}}\boldsymbol{W}_{\text{reflect}}\,d\boldsymbol{V} + D_{\boldsymbol{P}}\boldsymbol{W}_{\text{reflect}}\,d\boldsymbol{P} + D_{\boldsymbol{N}}\boldsymbol{W}_{\text{reflect}}\,d\boldsymbol{N} + D_\eta\boldsymbol{W}_{\text{reflect}}\,d\eta && \text{(11b)} \\ d\boldsymbol{W}_{\text{refract}} &= D_{\boldsymbol{V}}\boldsymbol{W}_{\text{refract}}\,d\boldsymbol{V} + D_{\boldsymbol{P}}\boldsymbol{W}_{\text{refract}}\,d\boldsymbol{P} + D_{\boldsymbol{N}}\boldsymbol{W}_{\text{refract}}\,d\boldsymbol{N} + D_\eta\boldsymbol{W}_{\text{refract}}\,d\eta && \text{(11c)} \end{aligned}$$

and subsequently identify the respective Jacobians.

Following [Igehy, 1999] we sketch an iterative process, where in each iteration: 1) The ray is transfered to the point of intersecting geometry, $\boldsymbol{Q}$, 2) the directions of reflection and refraction,$\boldsymbol{W}_{\text{reflect}}$ and $\boldsymbol{W}_{\text{refract}}$, are calculated simultaneously. The pairs $(\boldsymbol{Q}, \boldsymbol{W}_{\text{reflect}})$ and $(\boldsymbol{Q}, \boldsymbol{W}_{\text{refract}})$ are used as two new source points and directions for following iterations.

2.1 Transfer

The transfer of a ray onto a surface at distance s is,

$$\boldsymbol{Q} = \boldsymbol{P} + s\boldsymbol{V}. \tag{12}$$

We will assume that the surface is given implicitly as a scalar function $F : \mathbb{R}^3 \to \mathbb{R}$, where

$$0 = F(\boldsymbol{Q}), \tag{13}$$

and we will assume that there exists a method for solving for the smallest $s^* > 0$, where

$$0 = F(\boldsymbol{P} + s^*\boldsymbol{V}). \tag{14}$$

We require that the surface unit normal, $\boldsymbol{N}$, exists at $\boldsymbol{Q}$, and when F is smooth, then $\boldsymbol{N}$ is parallel to the spatial gradient of F. The differentialis found to be

$$d\boldsymbol{Q} = d\boldsymbol{P} + \boldsymbol{V}\, ds + s^*\, d\boldsymbol{V}. \tag{15}$$

If the differential ds depends on $d\boldsymbol{N}$, then ds will depend on the curvature of the surface at s^*. For convenience we will in the remainder of this article use the symbol s to denote s^*.

2.2 Reflection

Given a ray transferred to a surface, reflection is given by

$$\boldsymbol{W}_{\text{reflect}} = \boldsymbol{V} - 2(\boldsymbol{V}^T\boldsymbol{N})\boldsymbol{N}. \tag{16}$$

Hence,

$$\begin{aligned} d\boldsymbol{W}_{\text{reflect}} &= d\boldsymbol{V} - 2\left((d\boldsymbol{V}^T\,\boldsymbol{N} + \boldsymbol{V}^T d\boldsymbol{N})\boldsymbol{N} + (\boldsymbol{V}^T\boldsymbol{N})d\boldsymbol{N}\right) && \text{(17a)}\\ &= \left(\boldsymbol{I}_3 - 2\boldsymbol{N}\boldsymbol{N}^T\right) d\boldsymbol{V} - 2\left(\boldsymbol{V}^T\boldsymbol{N}\boldsymbol{I}_3 + \boldsymbol{N}\boldsymbol{V}^T\right) d\boldsymbol{N}. && \text{(17b)}\end{aligned}$$

2.3 Refraction

Given a ray transferred to a surface, refraction is given by Snell's law [Watt and Watt, 1992, Igehy, 1999],

$$\boldsymbol{W}_{\text{refract}} = \eta\boldsymbol{V} - \mu\boldsymbol{N}, \tag{18}$$

where

$$\begin{aligned} \mu &= \eta\boldsymbol{V}^T\boldsymbol{N} + \sqrt{\xi}, && \text{(19a)}\\ \xi &= 1 - \eta^2\left(1 - \left(\boldsymbol{V}^T\boldsymbol{N}\right)^2\right), && \text{(19b)}\end{aligned}$$

and η is the ratio of refraction indices of the material between the interfaced media. The refraction ratio between water and air is typically $\eta = 1.33$, which is why an often use approximation near $\eta = 1$ is $\xi \simeq \left(\boldsymbol{V}^T\boldsymbol{N}\right)^2$, nevertheless, we will derive the full structure to facilitate a greater range of η's.

The differential is found to be,

$$\begin{aligned} d\boldsymbol{W}_{\text{refract}} &= d\eta\,\boldsymbol{V} + \eta d\boldsymbol{V} - d\mu\,\boldsymbol{N} - \mu d\boldsymbol{N} && \text{(20a)}\\ &= \boldsymbol{V}d\eta + \eta d\boldsymbol{V} - \boldsymbol{N}d\mu - \mu d\boldsymbol{N}. && \text{(20b)}\end{aligned}$$

using $d\xi = -2\eta\left(1-\left(\boldsymbol{V}^T\boldsymbol{N}\right)^2\right)d\eta + 2\eta^2\left(\boldsymbol{V}^T\boldsymbol{N}\right)\left(d\boldsymbol{V}^T\boldsymbol{N} + \boldsymbol{V}^T d\boldsymbol{N}\right)$, we see that

$$
\begin{aligned}
d\mu &= \boldsymbol{V}^T\boldsymbol{N}d\eta + \eta(d\boldsymbol{V}^T\boldsymbol{N} + \boldsymbol{V}^T d\boldsymbol{N}) + \frac{d\xi}{2\sqrt{\xi}} && \text{(21a)}\\
&= \boldsymbol{V}^T\boldsymbol{N}d\eta + \eta(d\boldsymbol{V}^T\boldsymbol{N} + \boldsymbol{V}^T d\boldsymbol{N}) + \frac{-\eta\left(1-\left(\boldsymbol{V}^T\boldsymbol{N}\right)^2\right)d\eta + \eta^2\boldsymbol{V}^T\boldsymbol{N}(d\boldsymbol{V}^T\boldsymbol{N} + \boldsymbol{V}^T d\boldsymbol{N})}{\sqrt{\xi}} && \text{(21b)}\\
&= \left(\boldsymbol{V}^T\boldsymbol{N} - \frac{\eta\left(1-\left(\boldsymbol{V}^T\boldsymbol{N}\right)^2\right)}{\sqrt{\xi}}\right)d\eta + \eta(\boldsymbol{N}^T d\boldsymbol{V} + \boldsymbol{V}^T d\boldsymbol{N}) + \frac{\eta^2\boldsymbol{V}^T\boldsymbol{N}(\boldsymbol{N}^T d\boldsymbol{V} + \boldsymbol{V}^T d\boldsymbol{N})}{\sqrt{\xi}} && \text{(21c)}\\
&= \left(\boldsymbol{V}^T\boldsymbol{N} - \frac{1-\xi}{\eta\sqrt{\xi}}\right)d\eta + \eta\left(1 + \frac{\eta\boldsymbol{V}^T\boldsymbol{N}}{\sqrt{\xi}}\right)\boldsymbol{N}^T d\boldsymbol{V} + \eta\left(1 + \frac{\eta\boldsymbol{V}^T\boldsymbol{N}}{\sqrt{\xi}}\right)\boldsymbol{V}^T d\boldsymbol{N}. && \text{(21d)}
\end{aligned}
$$

Gathering terms we find that

$$
\begin{aligned}
d\boldsymbol{W}_{\text{refract}} = &\left(\boldsymbol{V} - \left(\boldsymbol{V}^T\boldsymbol{N} - \frac{1-\xi}{\eta\sqrt{\xi}}\right)\boldsymbol{N}\right)d\eta\\
&+ \left(\eta\boldsymbol{I}_3 - \eta\left(1 + \frac{\eta\boldsymbol{V}^T\boldsymbol{N}}{\sqrt{\xi}}\right)\boldsymbol{N}\boldsymbol{N}^T\right)d\boldsymbol{V}\\
&- \left(\mu\boldsymbol{I}_3 + \eta\left(1 + \frac{\eta\boldsymbol{V}^T\boldsymbol{N}}{\sqrt{\xi}}\right)\boldsymbol{N}\boldsymbol{V}^T\right)d\boldsymbol{N}. \qquad (22)
\end{aligned}
$$

3 Examples: Triangular Surface Models

A number of differentials described above depend on the surface of intersection. We will now evaluate the differentials to full depth for two popular and practical surface models based on triangles. The implicit function of the interior of a triangle is identical to that of a plane, and the implicit function of a plane with normal $\boldsymbol{N}_{\text{flat}}$, and where $\boldsymbol{Q}_0$ is a point in the plane, is given as

$$F(\boldsymbol{Q}) = (\boldsymbol{Q}_0 - \boldsymbol{Q})^T\boldsymbol{N}_{\text{flat}}. \qquad (23)$$

The models we will investigate originates from flat and Phong shading. The cumbersome "flat" subscript is used to distinguish the geometry normal from the interpolated normal in Phong shading. Further, we will assume that the media is homogeneous, i.e. $d\eta = 0$.

3.1 Flat Surface

Assuming that we have identified a triangle intersecting the ray, where $\boldsymbol{N}^T\boldsymbol{V} \neq 0$, then we combine (12) and (23) and seek the zero point,

$$0 = (\boldsymbol{Q}_0 - \boldsymbol{P} - s\boldsymbol{V})^T\boldsymbol{N}_{\text{flat}}, \qquad (24)$$

and

$$s = \frac{(\boldsymbol{Q}_0 - \boldsymbol{P})^T\boldsymbol{N}_{\text{flat}}}{\boldsymbol{V}^T\boldsymbol{N}_{\text{flat}}}. \qquad (25)$$

The full differential of ds is found as follows,

$$ds = \frac{\left(d\left(\left(\boldsymbol{Q}_0 - \boldsymbol{P}\right)^T \boldsymbol{N}_{\text{flat}}\right)\right)\left(\boldsymbol{V}^T \boldsymbol{N}_{\text{flat}}\right) - \left(\left(\boldsymbol{Q}_0 - \boldsymbol{P}\right)^T \boldsymbol{N}_{\text{flat}}\right) d\left(\boldsymbol{V}^T \boldsymbol{N}_{\text{flat}}\right)}{\left(\boldsymbol{V}^T \boldsymbol{N}_{\text{flat}}\right)^2} \tag{26a}$$

$$= \frac{\left(\left(d\boldsymbol{Q}_0 - d\boldsymbol{P}\right)^T \boldsymbol{N}_{\text{flat}} + \left(\boldsymbol{Q}_0 - \boldsymbol{P}\right)^T d\boldsymbol{N}_{\text{flat}}\right) - s\left(\left(d\boldsymbol{V}^T\right) \boldsymbol{N}_{\text{flat}} + \boldsymbol{V}^T d\boldsymbol{N}_{\text{flat}}\right)}{\boldsymbol{V}^T \boldsymbol{N}_{\text{flat}}} \tag{26b}$$

$$= \frac{\left(\boldsymbol{N}_{\text{flat}}^T \left(d\boldsymbol{Q}_0 - d\boldsymbol{P}\right) + \left(\boldsymbol{Q}_0 - \boldsymbol{P}\right)^T d\boldsymbol{N}_{\text{flat}}\right) - s\left(\boldsymbol{N}_{\text{flat}}^T d\boldsymbol{V} + \boldsymbol{V}^T d\boldsymbol{N}_{\text{flat}}\right)}{\boldsymbol{V}^T \boldsymbol{N}_{\text{flat}}} \tag{26c}$$

$$= \frac{\boldsymbol{N}_{\text{flat}}^T}{\boldsymbol{V}^T \boldsymbol{N}_{\text{flat}}} d\boldsymbol{Q}_0 - \frac{\boldsymbol{N}_{\text{flat}}^T}{\boldsymbol{V}^T \boldsymbol{N}_{\text{flat}}} d\boldsymbol{P} + \frac{\left(\boldsymbol{Q}_0 - \boldsymbol{P}\right)^T - s\boldsymbol{V}^T}{\boldsymbol{V}^T \boldsymbol{N}_{\text{flat}}} d\boldsymbol{N}_{\text{flat}} - \frac{s\boldsymbol{N}_{\text{flat}}^T}{\boldsymbol{V}^T \boldsymbol{N}_{\text{flat}}} d\boldsymbol{V}. \tag{26d}$$

Combining (15), (17), (22), and (26) we find,

$$d\boldsymbol{Q} = \boldsymbol{K} d\boldsymbol{P} + s\boldsymbol{K} d\boldsymbol{V} + \left(\boldsymbol{I}_3 - \boldsymbol{K}\right) d\boldsymbol{Q}_0 + \boldsymbol{L} d\boldsymbol{N}_{\text{flat}}, \tag{27a}$$

$$d\boldsymbol{W}_{\text{reflect}} = \left(\boldsymbol{I}_3 - 2\boldsymbol{N}_{\text{flat}}\boldsymbol{N}_{\text{flat}}^T\right) d\boldsymbol{V} - 2\left(\boldsymbol{V}^T \boldsymbol{N}_{\text{flat}} \boldsymbol{I}_3 + \boldsymbol{N}_{\text{flat}} \boldsymbol{V}^T\right) d\boldsymbol{N}_{\text{flat}}, \tag{27b}$$

$$d\boldsymbol{W}_{\text{refract}} = \left(\eta \boldsymbol{I}_3 - \eta\left(1 + \frac{\eta \boldsymbol{V}^T \boldsymbol{N}_{\text{flat}}}{\sqrt{\xi}}\right) \boldsymbol{N}_{\text{flat}} \boldsymbol{N}_{\text{flat}}^T\right) d\boldsymbol{V} - \left(\mu \boldsymbol{I}_3 + \eta\left(1 + \frac{\eta \boldsymbol{V}^T \boldsymbol{N}_{\text{flat}}}{\sqrt{\xi}}\right) \boldsymbol{N}_{\text{flat}} \boldsymbol{V}^T\right) d\boldsymbol{N}_{\text{flat}} \tag{27c}$$

where

$$\boldsymbol{K} = \boldsymbol{I}_3 - \frac{\boldsymbol{V}\boldsymbol{N}_{\text{flat}}^T}{\boldsymbol{V}^T \boldsymbol{N}_{\text{flat}}}, \tag{28a}$$

$$\boldsymbol{L} = \frac{\boldsymbol{V}\left(\boldsymbol{Q}_0 - \boldsymbol{P}\right)^T - s\boldsymbol{V}\boldsymbol{V}^T}{\boldsymbol{V}^T \boldsymbol{N}_{\text{flat}}}. \tag{28b}$$

Although the surface is flat, and the spatial part of $d\boldsymbol{N}_{\text{flat}}$ is zero, we cannot disregard terms involving $d\boldsymbol{N}_{\text{flat}}$, since the temporal part need not be zero.

Typically, a triangle will be parametrized by its 3 vertices, $\boldsymbol{Q}_0$, $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$, and a more natural parametrization of changes is in terms of the vertices. Such a parametrization allows us to further develop $d\boldsymbol{N}_{\text{flat}}$. Assume that,

$$\boldsymbol{n}_{\text{flat}} = (\boldsymbol{Q}_2 - \boldsymbol{Q}_0) \times (\boldsymbol{Q}_1 - \boldsymbol{Q}_0), \tag{29}$$

To be consistent w.r.t. models for reflection and refraction, we will assume that $\boldsymbol{n}^T\boldsymbol{V} < 0$, otherwise we will interchange $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$. For $\boldsymbol{n}^T\boldsymbol{V} < 0$ we find,

$$\boldsymbol{N}_{\text{flat}} = \frac{\boldsymbol{n}_{\text{flat}}}{\|\boldsymbol{n}_{\text{flat}}\|}, \tag{30a}$$

$$d\boldsymbol{N}_{\text{flat}} = \frac{\boldsymbol{n}_{\text{flat}}^T \boldsymbol{n}_{\text{flat}} \boldsymbol{I}_3 - \boldsymbol{n}_{\text{flat}} \boldsymbol{n}_{\text{flat}}^T}{(\boldsymbol{n}_{\text{flat}}^T \boldsymbol{n}_{\text{flat}})^{3/2}} d\boldsymbol{n}_{\text{flat}}, \tag{30b}$$

$$d\boldsymbol{n} = (d\boldsymbol{Q}_{2\times} - d\boldsymbol{Q}_{0\times})(\boldsymbol{Q}_1 - \boldsymbol{Q}_0) + (\boldsymbol{Q}_{2\times} - \boldsymbol{Q}_{0\times})(d\boldsymbol{Q}_1 - d\boldsymbol{Q}_0) \tag{30c}$$

$$= (\boldsymbol{Q}_{2\times} - \boldsymbol{Q}_{0\times})(d\boldsymbol{Q}_1 - d\boldsymbol{Q}_0) - (\boldsymbol{Q}_{1\times} - \boldsymbol{Q}_{0\times})(d\boldsymbol{Q}_2 - d\boldsymbol{Q}_0) \tag{30d}$$

$$= (\boldsymbol{Q}_{0\times} - \boldsymbol{Q}_{1\times})d\boldsymbol{Q}_2 + (\boldsymbol{Q}_{2\times} - \boldsymbol{Q}_{0\times})d\boldsymbol{Q}_1 + (\boldsymbol{Q}_{1\times} - \boldsymbol{Q}_{2\times})d\boldsymbol{Q}_0. \tag{30e}$$

For simplicity we have used the matrix form of cross products, $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a}_\times \boldsymbol{b} = \boldsymbol{b}_\times^T \boldsymbol{a} = -\boldsymbol{b}_\times \boldsymbol{a}$, where

$$\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow \boldsymbol{c}_\times = \begin{bmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{bmatrix}, \tag{31}$$

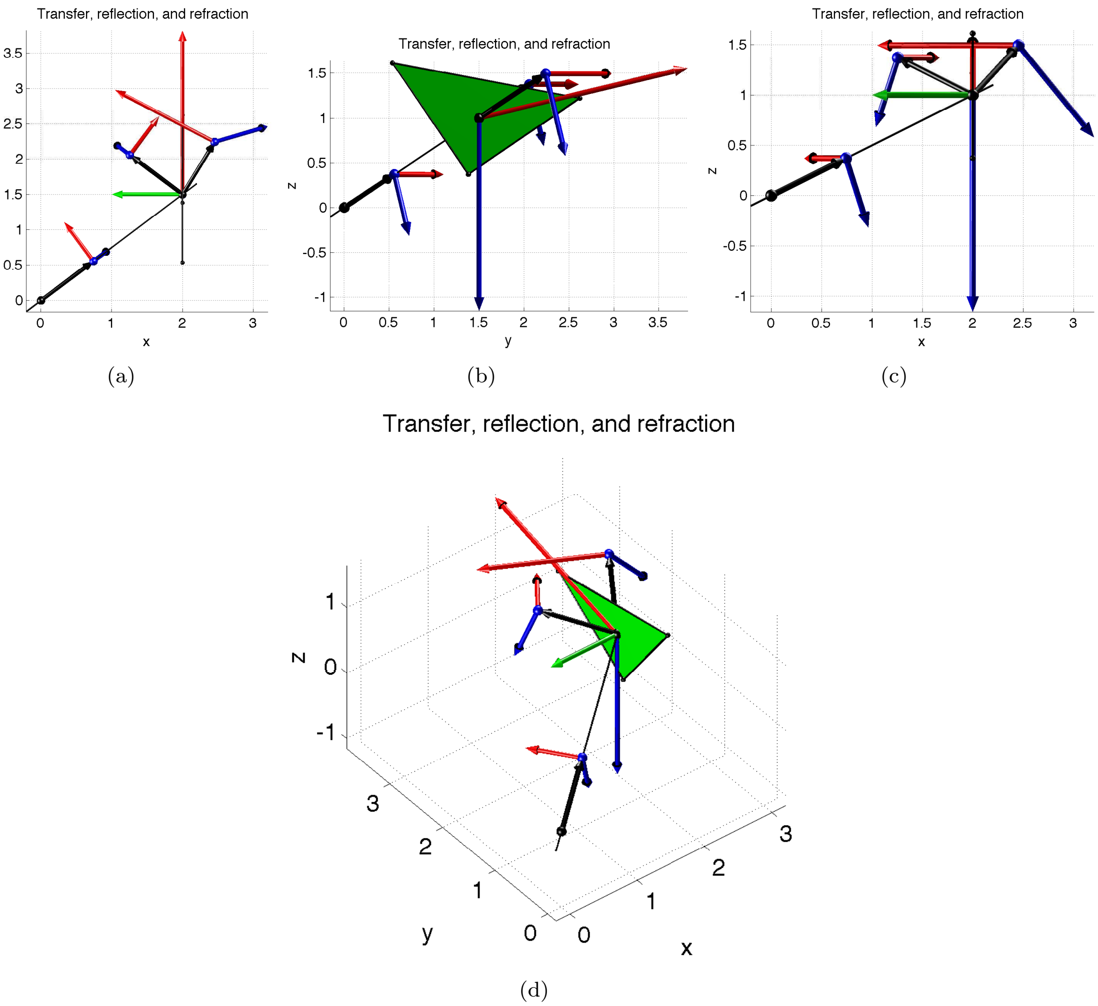

Figure 2: Transfer, Reflection, and Refraction for Flat surfaces. Black arrows are ray directions, green is triangle normal, blue and red arrow illustrate the row vectors of $d\boldsymbol{P}/d\boldsymbol{\theta}$, $d\boldsymbol{Q}/d\boldsymbol{\theta}$, $d\boldsymbol{V}/d\boldsymbol{\theta}$, and $d\boldsymbol{W}/d\boldsymbol{\theta}$ as relevant. Subfigures (a)-(c) show orthographic projections of (d).

Using

$$\boldsymbol{J} = \frac{\boldsymbol{n}_{\text{flat}}^T \boldsymbol{n}_{\text{flat}} \boldsymbol{I}_3 - \boldsymbol{n}_{\text{flat}} \boldsymbol{n}_{\text{flat}}^T}{(\boldsymbol{n}_{\text{flat}}^T \boldsymbol{n}_{\text{flat}})^{3/2}}, \tag{32}$$

we find that

$$d\boldsymbol{Q} = \boldsymbol{K}d\boldsymbol{P} + s\boldsymbol{K}d\boldsymbol{V} + (\boldsymbol{I}_3 - \boldsymbol{K} + \boldsymbol{L}\boldsymbol{J}(\boldsymbol{Q}_{1\times} - \boldsymbol{Q}_{2\times}))\, d\boldsymbol{Q}_0 + \boldsymbol{L}\boldsymbol{J}(\boldsymbol{Q}_{2\times} - \boldsymbol{Q}_{0\times})d\boldsymbol{Q}_1 + \boldsymbol{L}\boldsymbol{J}(\boldsymbol{Q}_{0\times} - \boldsymbol{Q}_{1\times})d\boldsymbol{Q}_2, \tag{33}$$

For stationary, flat surfaces $d\boldsymbol{N}_{\text{flat}} = 0$ and $d\boldsymbol{Q}_i = 0, i = 0 \ldots 2$, and we may write $d\boldsymbol{Q} = \boldsymbol{K}d\boldsymbol{P} + s\boldsymbol{K}d\boldsymbol{V}$ in agreement with [Igehy, 1999]. The rays and spatial differentials are illustrated in Figure 2. In Figure 3 are examples of time differentials shown. The yellow arrows denote velocity vectors, and in Figures 3(a)-(c) it should be noted, that a velocity of $\boldsymbol{P}$ in different directions implies a velocity of $\boldsymbol{Q}$ in the plane of the triangle. In Figures 3(d)-(e) we see, that a rotational velocity of $\boldsymbol{V}$ implies both a velocity of $\boldsymbol{Q}$ in the plane of the triangle as well as a rotation of $\boldsymbol{W}_{\text{reflect}}$ and $\boldsymbol{W}_{\text{refract}}$. Finally but

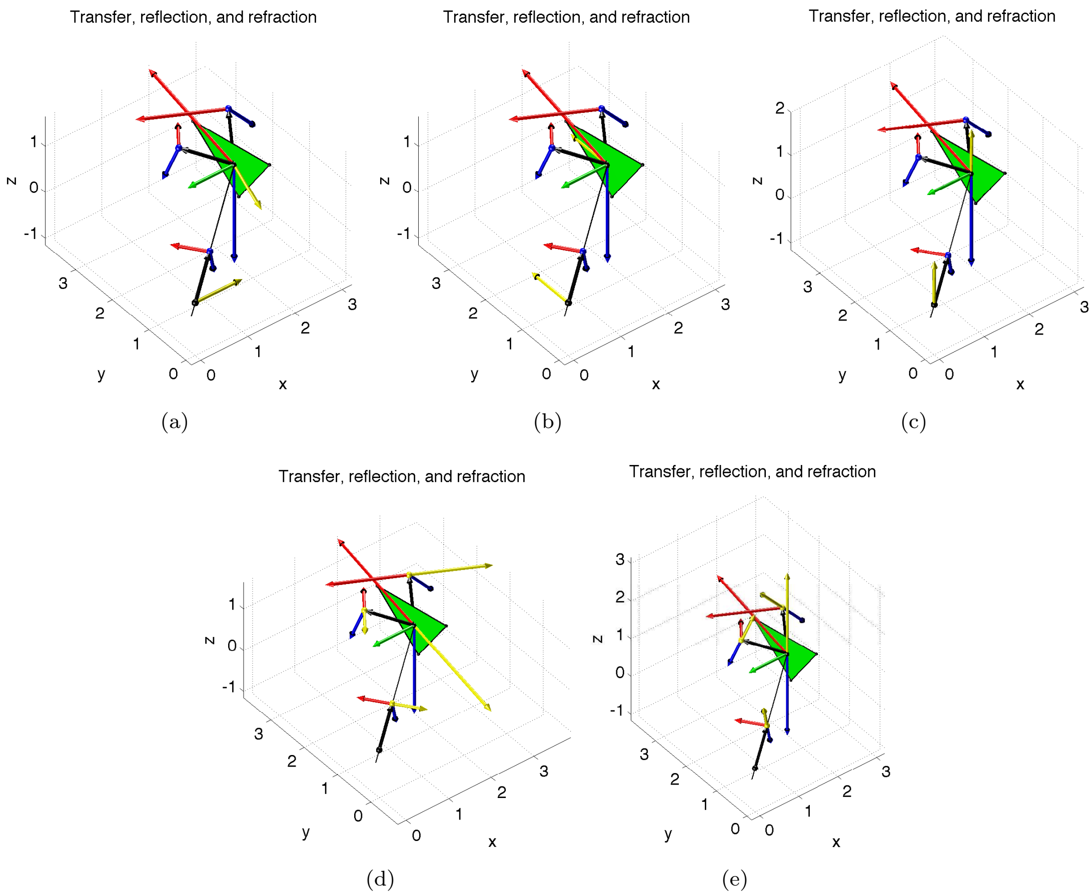

Figure 3: Time differentials for Flat surfaces. Yellow arrows denote imposed and resulting time derivatives. Subfigures (a)-(c) shows imposed velocities in three orthogonal directions on the origin, $\boldsymbol{P}$, (d)-(e) shows imposed rotational velocities in viewing direction $\boldsymbol{V}$, and (f)-(h) shows imposed velocities in three orthogonal directions on one of the vertices.

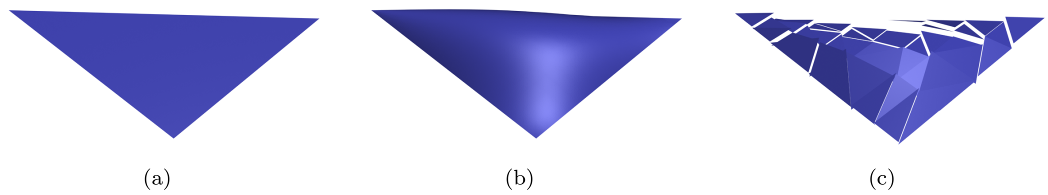

Figure 4: Phong shading assumes fish scale geometry. A triangle, 4(a), shaded with Phong's model, 4(b), expresses a complexity not supported by the real geometry. One way of conceptualizing this model is to think of the triangle as consisting of fish scales, 4(c); in this mindset every point on the surface of the triangle is associated with an independent local plane or fish scale whose normal is interpolated from the corners of the triangle.

not shown, motion of the triangle normal to the triangle normal does not imply any velocity on any parameters, and a rotation of the triangle normal implies a velocity on $\boldsymbol{Q}$ along the ray.

3.2 Phong Shaded Surface

Phong shading uses a triangle as a base geometry but imposes varying normals across it. Since the flatness of the triangle contradicts the changing normals, we prefer to think of this as a fish scale model as illustrated in Figure 4.

Phong shading assumes a plane represented by the 3 vertices of a triangle, $\boldsymbol{Q}_0$, $\boldsymbol{Q}_1$, and $\boldsymbol{Q}_2$, and corresponding vertex normals $\boldsymbol{N}_0$, $\boldsymbol{N}_1$, and $\boldsymbol{N}_2$. To calculate the intersection of the view ray with the triangle we use the Flat surface model (23), calculate the flat normal, $\boldsymbol{N}_{\text{flat}}$, by (29), and we find the point of intersection by solving (25). For reflection and refraction we construct an linearly interpolated normal from the three vertex normals. The flat normal and interpolated vertex normal most often will not coincide, and as a consequence $d\boldsymbol{Q}/d\boldsymbol{N}_{\text{phong}}$ will not span the triangle. Therefore, we calculate $d\boldsymbol{Q}$ by (33).

To interpolate the vertex normals at the point of intersection, $\boldsymbol{Q}$, we calculate the Barycentric coordinates,

$$\boldsymbol{Q} = \lambda_0 \boldsymbol{Q}_0 + \lambda_1 \boldsymbol{Q}_1 + \lambda_2 \boldsymbol{Q}_2, \tag{34}$$

where $\lambda_i \geq 0$ are homogeneous Barycentric coordinates such that $\lambda_0 + \lambda_1 + \lambda_2 = 1$. The Barycentric coordinates are then used to interpolate the vertex normals as,

$$\boldsymbol{n} = \lambda_0 \boldsymbol{N}_0 + \lambda_1 \boldsymbol{N}_1 + \lambda_2 \boldsymbol{N}_2, \tag{35a}$$

$$\boldsymbol{N} = \frac{\boldsymbol{n}}{\|\boldsymbol{n}\|}. \tag{35b}$$

Note that the Barycentric coordinates are local to the triangle, and their differentials $d\lambda_i$ may be used to estimate the change in $\boldsymbol{Q}$ in terms of the triangle. However, this requires algorithmic care near the border of the triangle, where $\lambda_i + \Delta\lambda_i$ may fall outside the triangle.

Assuming a ray passing through $\boldsymbol{P}$ with direction $\boldsymbol{V}$, which intersects a triangle within vertices $\boldsymbol{Q}_0$, $\boldsymbol{Q}_1$, and $\boldsymbol{Q}_2$, then $0 \leq \lambda_i \leq 1$, and we may find the Barycentric coordinates using Möller and Trumbore's algorithm [Möller and Trumbore, 1997]: Let

$$\boldsymbol{E}_0 = \boldsymbol{Q}_1 - \boldsymbol{Q}_0, \tag{36a}$$

$$\boldsymbol{E}_1 = \boldsymbol{Q}_2 - \boldsymbol{Q}_0, \tag{36b}$$

$$\boldsymbol{T} = \boldsymbol{P} - \boldsymbol{Q}_0 \tag{36c}$$

then

$$\lambda_1 = \frac{(\boldsymbol{V} \times \boldsymbol{T})^T \boldsymbol{E}_1}{(\boldsymbol{V} \times \boldsymbol{E}_0)^T \boldsymbol{E}_1} \tag{37a}$$

$$= \frac{\boldsymbol{\tau}^T \boldsymbol{E}_1}{\boldsymbol{\gamma}^T \boldsymbol{E}_1} \tag{37b}$$

$$\lambda_2 = \frac{(\boldsymbol{V} \times \boldsymbol{T})^T \boldsymbol{E}_0}{(\boldsymbol{V} \times \boldsymbol{E}_1)^T \boldsymbol{E}_0} \tag{37c}$$

$$= \frac{\boldsymbol{\tau}^T \boldsymbol{E}_0}{\boldsymbol{\zeta}^T \boldsymbol{E}_0}, \tag{37d}$$

where $\boldsymbol{\tau} = \boldsymbol{V} \times \boldsymbol{T}$, $\boldsymbol{\gamma} = \boldsymbol{V} \times \boldsymbol{E}_0$, $\boldsymbol{\zeta} = \boldsymbol{V} \times \boldsymbol{E}_1$, and $\lambda_0 = 1 - \lambda_1 - \lambda_2$. The differential, $d\boldsymbol{N}$, is now found to be,

$$d\boldsymbol{N} = \frac{\boldsymbol{n}^T \boldsymbol{n} \boldsymbol{I}_3 - \boldsymbol{n}\boldsymbol{n}^T}{(\boldsymbol{n}^T \boldsymbol{n})^{3/2}} d\boldsymbol{n}, \tag{38a}$$

$$d\boldsymbol{n} = \boldsymbol{N}_0 d\lambda_0 + \lambda_0 d\boldsymbol{N}_0 + \boldsymbol{N}_1 d\lambda_1 + \lambda_1 d\boldsymbol{N}_1 + \boldsymbol{N}_2 d\lambda_2 + \lambda_2 d\boldsymbol{N}_2. \tag{38b}$$

Since,

$$d\lambda_0 = -d\lambda_1 - d\lambda_2, \tag{39a}$$

$$\begin{aligned} d\lambda_1 &= \frac{(d\boldsymbol{\tau}^T \boldsymbol{E}_1 + \boldsymbol{\tau}^T d\boldsymbol{E}_1)\boldsymbol{\gamma}^T \boldsymbol{E}_1 - \boldsymbol{\tau}^T \boldsymbol{E}_1 (d\boldsymbol{\gamma}^T \boldsymbol{E}_1 + \boldsymbol{\gamma}^T d\boldsymbol{E}_1)}{(\boldsymbol{\gamma}^T \boldsymbol{E}_1)^2} \\ &= \frac{\boldsymbol{\gamma}^T \boldsymbol{E}_1 (\boldsymbol{E}_1^T d\boldsymbol{\tau} + \boldsymbol{\tau}^T d\boldsymbol{E}_1) - \boldsymbol{\tau}^T \boldsymbol{E}_1 (\boldsymbol{E}_1^T d\boldsymbol{\gamma} + \boldsymbol{\gamma}^T d\boldsymbol{E}_1)}{(\boldsymbol{\gamma}^T \boldsymbol{E}_1)^2} \\ &= \frac{\boldsymbol{\gamma}^T \boldsymbol{S}_1 d\boldsymbol{\tau} - \boldsymbol{\tau}^T \boldsymbol{S}_1 d\boldsymbol{\gamma} + \boldsymbol{E}_1^T \left(\boldsymbol{\gamma}\boldsymbol{\tau}^T - \boldsymbol{\tau}\boldsymbol{\gamma}^T\right) d\boldsymbol{E}_1}{\boldsymbol{\gamma}^T \boldsymbol{S}_1 \boldsymbol{\gamma}}, \end{aligned} \tag{39b}$$

$$\begin{aligned} d\lambda_2 &= \frac{(d\boldsymbol{\tau}^T \boldsymbol{E}_0 + \boldsymbol{\tau}^T d\boldsymbol{E}_0)\boldsymbol{\zeta}^T \boldsymbol{E}_0 - \boldsymbol{\tau}^T \boldsymbol{E}_0 (d\boldsymbol{\zeta}^T \boldsymbol{E}_0 + \boldsymbol{\zeta}^T d\boldsymbol{E}_0)}{(\boldsymbol{\zeta}^T \boldsymbol{E}_0)^2} \\ &= \frac{\boldsymbol{\zeta}^T \boldsymbol{E}_0 (\boldsymbol{E}_0^T d\boldsymbol{\tau} + \boldsymbol{\tau}^T d\boldsymbol{E}_0) - \boldsymbol{\tau}^T \boldsymbol{E}_0 (\boldsymbol{E}_0^T d\boldsymbol{\zeta} + \boldsymbol{\zeta}^T d\boldsymbol{E}_0)}{(\boldsymbol{\zeta}^T \boldsymbol{E}_0)^2} \\ &= \frac{\boldsymbol{\zeta}^T \boldsymbol{S}_0 d\boldsymbol{\tau} - \boldsymbol{\tau}^T \boldsymbol{S}_0 d\boldsymbol{\zeta} + \boldsymbol{E}_0^T \left(\boldsymbol{\zeta}\boldsymbol{\tau}^T - \boldsymbol{\tau}\boldsymbol{\zeta}^T\right) d\boldsymbol{E}_0}{\boldsymbol{\zeta}^T \boldsymbol{S}_0 \zeta}, \end{aligned} \tag{39c}$$

where $\boldsymbol{S}_1 = \boldsymbol{E}_1 \boldsymbol{E}_1^T$, and $\boldsymbol{S}_0 = \boldsymbol{E}_0 \boldsymbol{E}_0^T$. Thus we find that

$$\begin{aligned} d\boldsymbol{N} = \boldsymbol{J}\Bigg(& \lambda_0 d\boldsymbol{N}_0 + \lambda_1 d\boldsymbol{N}_1 + \lambda_2 d\boldsymbol{N}_2 \\ & + (\boldsymbol{N}_1 - \boldsymbol{N}_0) \left(\frac{\boldsymbol{\gamma}^T \boldsymbol{S}_1 d\boldsymbol{\tau} - \boldsymbol{\tau}^T \boldsymbol{S}_1 d\boldsymbol{\gamma} + \boldsymbol{E}_1^T \left(\boldsymbol{\gamma}\boldsymbol{\tau}^T - \boldsymbol{\tau}\boldsymbol{\gamma}^T\right) d\boldsymbol{E}_1}{\boldsymbol{\gamma}^T \boldsymbol{S}_1 \boldsymbol{\gamma}} \right) \\ & + (\boldsymbol{N}_2 - \boldsymbol{N}_0) \left(\frac{\boldsymbol{\zeta}^T \boldsymbol{S}_0 d\boldsymbol{\tau} - \boldsymbol{\tau}^T \boldsymbol{S}_0 d\boldsymbol{\zeta} + \boldsymbol{E}_0^T \left(\boldsymbol{\zeta}\boldsymbol{\tau}^T - \boldsymbol{\tau}\boldsymbol{\zeta}^T\right) d\boldsymbol{E}_0}{\boldsymbol{\zeta}^T \boldsymbol{S}_0 \zeta} \right) \Bigg) \end{aligned} \tag{40a}$$

$$\begin{aligned} = \boldsymbol{J}\Big(& \lambda_0 d\boldsymbol{N}_0 + \lambda_1 d\boldsymbol{N}_1 + \lambda_2 d\boldsymbol{N}_2 \\ & + \left(\boldsymbol{\Delta}_1 \boldsymbol{\gamma}^T \boldsymbol{S}_1 + \boldsymbol{\Delta}_2 \boldsymbol{\zeta}^T \boldsymbol{S}_0\right) d\boldsymbol{\tau} - \boldsymbol{\Delta}_1 \boldsymbol{\tau}^T \boldsymbol{S}_1 d\boldsymbol{\gamma} - \boldsymbol{\Delta}_2 \boldsymbol{\tau}^T \boldsymbol{S}_0 d\boldsymbol{\zeta} \\ & + \boldsymbol{\Delta}_2 \boldsymbol{E}_0^T \left(\boldsymbol{\zeta}\boldsymbol{\tau}^T - \boldsymbol{\tau}\boldsymbol{\zeta}^T\right) d\boldsymbol{E}_0 + \boldsymbol{\Delta}_1 \boldsymbol{E}_1^T \left(\boldsymbol{\gamma}\boldsymbol{\tau}^T - \boldsymbol{\tau}\boldsymbol{\gamma}^T\right) d\boldsymbol{E}_1 \Big) \end{aligned} \tag{40b}$$

where $\boldsymbol{J} = \frac{\boldsymbol{n}^T \boldsymbol{n} \boldsymbol{I}_3 - \boldsymbol{n}\boldsymbol{n}^T}{(\boldsymbol{n}^T \boldsymbol{n})^{3/2}}$, $\boldsymbol{\Delta}_1 = \frac{(\boldsymbol{N}_1 - \boldsymbol{N}_0)}{\boldsymbol{\gamma}^T \boldsymbol{S}_1 \boldsymbol{\gamma}}$, and $\boldsymbol{\Delta}_2 = \frac{(\boldsymbol{N}_2 - \boldsymbol{N}_0)}{\boldsymbol{\zeta}^T \boldsymbol{S}_0 \zeta}$. For simplicity we convert cross products into matrix form, $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a}_\times \boldsymbol{b} = \boldsymbol{b}_\times^T \boldsymbol{a} = -\boldsymbol{b}_\times \boldsymbol{a}$, where

$$\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow \boldsymbol{c}_\times = \begin{bmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{bmatrix}, \tag{41}$$

hence,

$$d\boldsymbol{\tau} = d\boldsymbol{V}_\times \boldsymbol{T} + \boldsymbol{V}_\times d\boldsymbol{T} = \boldsymbol{V}_\times d\boldsymbol{T} - \boldsymbol{T}_\times d\boldsymbol{V}, \tag{42a}$$
$$d\boldsymbol{\gamma} = d\boldsymbol{V}_\times \boldsymbol{E}_0 + \boldsymbol{V}_\times d\boldsymbol{E}_0 = \boldsymbol{V}_\times d\boldsymbol{E}_0 - \boldsymbol{E}_{0\times} d\boldsymbol{V}, \tag{42b}$$
$$d\boldsymbol{\zeta} = d\boldsymbol{V}_\times \boldsymbol{E}_1 + \boldsymbol{V}_\times d\boldsymbol{E}_1 = \boldsymbol{V}_\times d\boldsymbol{E}_1 - \boldsymbol{E}_{1\times} d\boldsymbol{V}, \tag{42c}$$

implying that

$$\begin{aligned} d\boldsymbol{N} = \boldsymbol{J}\Big(& \lambda_0 d\boldsymbol{N}_0 + \lambda_1 d\boldsymbol{N}_1 + \lambda_2 d\boldsymbol{N}_2 \\ & + \left(\boldsymbol{\Delta}_1 \left(\boldsymbol{\tau}^T \boldsymbol{S}_1 \boldsymbol{E}_{0\times} - \boldsymbol{\gamma}^T \boldsymbol{S}_1 \boldsymbol{T}_\times\right) + \boldsymbol{\Delta}_2 \left(\boldsymbol{\tau}^T \boldsymbol{S}_0 \boldsymbol{E}_{1\times} - \boldsymbol{\zeta}^T \boldsymbol{S}_0 \boldsymbol{T}_\times\right)\right) d\boldsymbol{V} \\ & + \left(\boldsymbol{\Delta}_1 \boldsymbol{\gamma}^T \boldsymbol{S}_1 + \boldsymbol{\Delta}_2 \boldsymbol{\zeta}^T \boldsymbol{S}_0\right) \boldsymbol{V}_\times d\boldsymbol{T} \\ & + \left(\boldsymbol{\Delta}_2 \boldsymbol{E}_0^T \left(\boldsymbol{\zeta}\boldsymbol{\tau}^T - \boldsymbol{\tau}\boldsymbol{\zeta}^T\right) - \boldsymbol{\Delta}_1 \boldsymbol{\tau}^T \boldsymbol{S}_1 \boldsymbol{V}_\times\right) d\boldsymbol{E}_0 \\ & + \left(\boldsymbol{\Delta}_1 \boldsymbol{E}_1^T \left(\boldsymbol{\gamma}\boldsymbol{\tau}^T - \boldsymbol{\tau}\boldsymbol{\gamma}^T\right) - \boldsymbol{\Delta}_2 \boldsymbol{\tau}^T \boldsymbol{S}_0 \boldsymbol{V}_\times\right) d\boldsymbol{E}_1 \Big). \end{aligned} \tag{43}$$

Since $d\boldsymbol{E}_0 = d\boldsymbol{Q}_1 - d\boldsymbol{Q}_0$, $d\boldsymbol{E}_1 = d\boldsymbol{Q}_2 - d\boldsymbol{Q}_0$, $d\boldsymbol{T} = d\boldsymbol{P} - d\boldsymbol{Q}_0$, we find that

$$\begin{aligned} d\boldsymbol{N} = \boldsymbol{J}\Big(& \lambda_0 d\boldsymbol{N}_0 + \lambda_1 d\boldsymbol{N}_1 + \lambda_2 d\boldsymbol{N}_2 \\ & + \left(\boldsymbol{\Delta}_1 \left(\boldsymbol{\tau}^T \boldsymbol{S}_1 \boldsymbol{E}_{0\times} - \boldsymbol{\gamma}^T \boldsymbol{S}_1 \boldsymbol{T}_\times\right) + \boldsymbol{\Delta}_2 \left(\boldsymbol{\tau}^T \boldsymbol{S}_0 \boldsymbol{E}_{1\times} - \boldsymbol{\zeta}^T \boldsymbol{S}_0 \boldsymbol{T}_\times\right)\right) d\boldsymbol{V} \\ & + \left(\boldsymbol{\Delta}_1 \boldsymbol{\gamma}^T \boldsymbol{S}_1 + \boldsymbol{\Delta}_2 \boldsymbol{\zeta}^T \boldsymbol{S}_0\right) \boldsymbol{V}_\times d\boldsymbol{P} \\ & + \left(\boldsymbol{\Delta}_1 \left(\boldsymbol{\tau}^T - \boldsymbol{\gamma}^T\right) \boldsymbol{S}_1 \boldsymbol{V}_\times + \boldsymbol{\Delta}_2 \left(\boldsymbol{\tau}^T - \boldsymbol{\zeta}^T\right) \boldsymbol{S}_0 \boldsymbol{V}_\times + \boldsymbol{\Delta}_2 \boldsymbol{\Xi}_0 + \boldsymbol{\Delta}_1 \boldsymbol{\Xi}_1\right) d\boldsymbol{Q}_0 \\ & - \left(\boldsymbol{\Delta}_2 \boldsymbol{\Xi}_0 + \boldsymbol{\Delta}_1 \boldsymbol{\tau}^T \boldsymbol{S}_1 \boldsymbol{V}_\times\right) d\boldsymbol{Q}_1 \\ & - \left(\boldsymbol{\Delta}_1 \boldsymbol{\Xi}_1 + \boldsymbol{\Delta}_2 \boldsymbol{\tau}^T \boldsymbol{S}_0 \boldsymbol{V}_\times\right) d\boldsymbol{Q}_2 \Big) \end{aligned} \tag{44}$$

where $\boldsymbol{\Xi}_0 = \boldsymbol{E}_0^T \left(\boldsymbol{\tau}\boldsymbol{\zeta}^T - \boldsymbol{\zeta}\boldsymbol{\tau}^T\right)$, and $\boldsymbol{\Xi}_1 = \boldsymbol{E}_1^T \left(\boldsymbol{\tau}\boldsymbol{\gamma}^T - \boldsymbol{\gamma}\boldsymbol{\tau}^T\right)$. Gathering terms for reflection and refraction we find that

$$
\begin{aligned}
d\boldsymbol{W}_{\text{reflect}} &= \left(\boldsymbol{I}_3 - 2\boldsymbol{N}\boldsymbol{N}^T\right) d\boldsymbol{V} - 2\left(\boldsymbol{V}^T\boldsymbol{N}\boldsymbol{I}_3 + \boldsymbol{N}\boldsymbol{V}^T\right) d\boldsymbol{N} && \text{(45a)}\\
&= \left(\boldsymbol{I}_3 - 2\boldsymbol{N}\boldsymbol{N}^T\right) d\boldsymbol{V}\\
&\quad + \boldsymbol{M}\Bigg(\lambda_0 d\boldsymbol{N}_0 + \lambda_1 d\boldsymbol{N}_1 + \lambda_2 d\boldsymbol{N}_2\\
&\qquad + \left(\boldsymbol{\Delta}_1\left(\boldsymbol{\tau}^T\boldsymbol{S}_1\boldsymbol{E}_{0\times} - \boldsymbol{\gamma}^T\boldsymbol{S}_1\boldsymbol{T}_\times\right) + \boldsymbol{\Delta}_2\left(\boldsymbol{\tau}^T\boldsymbol{S}_0\boldsymbol{E}_{1\times} - \boldsymbol{\zeta}^T\boldsymbol{S}_0\boldsymbol{T}_\times\right)\right) d\boldsymbol{V}\\
&\qquad + \left(\boldsymbol{\Delta}_1\boldsymbol{\gamma}^T\boldsymbol{S}_1 + \boldsymbol{\Delta}_2\boldsymbol{\zeta}^T\boldsymbol{S}_0\right)\boldsymbol{V}_\times d\boldsymbol{P}\\
&\qquad + \left(\boldsymbol{\Delta}_1\left(\boldsymbol{\tau}^T - \boldsymbol{\gamma}^T\right)\boldsymbol{S}_1\boldsymbol{V}_\times + \boldsymbol{\Delta}_2\left(\boldsymbol{\tau}^T - \boldsymbol{\zeta}^T\right)\boldsymbol{S}_0\boldsymbol{V}_\times + \boldsymbol{\Delta}_2\boldsymbol{\Xi}_0 + \boldsymbol{\Delta}_1\boldsymbol{\Xi}_1\right) d\boldsymbol{Q}_0\\
&\qquad - \left(\boldsymbol{\Delta}_2\boldsymbol{\Xi}_0 + \boldsymbol{\Delta}_1\boldsymbol{\tau}^T\boldsymbol{S}_1\boldsymbol{V}_\times\right) d\boldsymbol{Q}_1\\
&\qquad - \left(\boldsymbol{\Delta}_1\boldsymbol{\Xi}_1 + \boldsymbol{\Delta}_2\boldsymbol{\tau}^T\boldsymbol{S}_0\boldsymbol{V}_\times\right) d\boldsymbol{Q}_2\Bigg). && \text{(45b)}\\
&= \left(\boldsymbol{I}_3 - 2\boldsymbol{N}\boldsymbol{N}^T + \boldsymbol{M}\left(\boldsymbol{\Delta}_1\left(\boldsymbol{\tau}^T\boldsymbol{S}_1\boldsymbol{E}_{0\times} - \boldsymbol{\gamma}^T\boldsymbol{S}_1\boldsymbol{T}_\times\right) + \boldsymbol{\Delta}_2\left(\boldsymbol{\tau}^T\boldsymbol{S}_0\boldsymbol{E}_{1\times} - \boldsymbol{\zeta}^T\boldsymbol{S}_0\boldsymbol{T}_\times\right)\right)\right) d\boldsymbol{V}\\
&\quad + \boldsymbol{M}\lambda_0 d\boldsymbol{N}_0 + \boldsymbol{M}\lambda_1 d\boldsymbol{N}_1 + \boldsymbol{M}\lambda_2 d\boldsymbol{N}_2\\
&\quad + \boldsymbol{M}\left(\boldsymbol{\Delta}_1\boldsymbol{\gamma}^T\boldsymbol{S}_1 + \boldsymbol{\Delta}_2\boldsymbol{\zeta}^T\boldsymbol{S}_0\right)\boldsymbol{V}_\times d\boldsymbol{P}\\
&\quad + \boldsymbol{M}\left(\boldsymbol{\Delta}_1\left(\boldsymbol{\tau}^T - \boldsymbol{\gamma}^T\right)\boldsymbol{S}_1\boldsymbol{V}_\times + \boldsymbol{\Delta}_2\left(\boldsymbol{\tau}^T - \boldsymbol{\zeta}^T\right)\boldsymbol{S}_0\boldsymbol{V}_\times + \boldsymbol{\Delta}_2\boldsymbol{\Xi}_0 + \boldsymbol{\Delta}_1\boldsymbol{\Xi}_1\right) d\boldsymbol{Q}_0\\
&\quad - \boldsymbol{M}\left(\boldsymbol{\Delta}_2\boldsymbol{\Xi}_0 + \boldsymbol{\Delta}_1\boldsymbol{\tau}^T\boldsymbol{S}_1\boldsymbol{V}_\times\right) d\boldsymbol{Q}_1\\
&\quad - \boldsymbol{M}\left(\boldsymbol{\Delta}_1\boldsymbol{\Xi}_1 + \boldsymbol{\Delta}_2\boldsymbol{\tau}^T\boldsymbol{S}_0\boldsymbol{V}_\times\right) d\boldsymbol{Q}_2. && \text{(45c)}
\end{aligned}
$$

where $\boldsymbol{M} = -2\left(\boldsymbol{V}^T\boldsymbol{N}\boldsymbol{I}_3 + \boldsymbol{N}\boldsymbol{V}^T\right)\boldsymbol{J}$, and

$$
d\boldsymbol{W}_{\text{refract}} = \left(\eta\boldsymbol{I}_3 - \eta\left(1 + \frac{\eta\boldsymbol{V}^T\boldsymbol{N}}{\sqrt{\xi}}\right)\boldsymbol{N}\boldsymbol{N}^T\right) d\boldsymbol{V} - \left(\mu\boldsymbol{I}_3 + \eta\left(1 + \frac{\eta\boldsymbol{V}^T\boldsymbol{N}}{\sqrt{\xi}}\right)\boldsymbol{N}\boldsymbol{V}^T\right) d\boldsymbol{N} \tag{46}
$$

$$
\begin{aligned}
&= \left(\eta\boldsymbol{I}_3 - \eta\left(1 + \frac{\eta\boldsymbol{V}^T\boldsymbol{N}}{\sqrt{\xi}}\right)\boldsymbol{N}\boldsymbol{N}^T\right) d\boldsymbol{V}\\
&\quad + \boldsymbol{H}\Bigg(\lambda_0 d\boldsymbol{N}_0 + \lambda_1 d\boldsymbol{N}_1 + \lambda_2 d\boldsymbol{N}_2\\
&\qquad + \left(\boldsymbol{\Delta}_1\left(\boldsymbol{\tau}^T\boldsymbol{S}_1\boldsymbol{E}_{0\times} - \boldsymbol{\gamma}^T\boldsymbol{S}_1\boldsymbol{T}_\times\right) + \boldsymbol{\Delta}_2\left(\boldsymbol{\tau}^T\boldsymbol{S}_0\boldsymbol{E}_{1\times} - \boldsymbol{\zeta}^T\boldsymbol{S}_0\boldsymbol{T}_\times\right)\right) d\boldsymbol{V}\\
&\qquad + \left(\boldsymbol{\Delta}_1\boldsymbol{\gamma}^T\boldsymbol{S}_1 + \boldsymbol{\Delta}_2\boldsymbol{\zeta}^T\boldsymbol{S}_0\right)\boldsymbol{V}_\times d\boldsymbol{P}\\
&\qquad + \left(\boldsymbol{\Delta}_1\left(\boldsymbol{\tau}^T - \boldsymbol{\gamma}^T\right)\boldsymbol{S}_1\boldsymbol{V}_\times + \boldsymbol{\Delta}_2\left(\boldsymbol{\tau}^T - \boldsymbol{\zeta}^T\right)\boldsymbol{S}_0\boldsymbol{V}_\times + \boldsymbol{\Delta}_2\boldsymbol{\Xi}_0 + \boldsymbol{\Delta}_1\boldsymbol{\Xi}_1\right) d\boldsymbol{Q}_0\\
&\qquad - \left(\boldsymbol{\Delta}_2\boldsymbol{\Xi}_0 + \boldsymbol{\Delta}_1\boldsymbol{\tau}^T\boldsymbol{S}_1\boldsymbol{V}_\times\right) d\boldsymbol{Q}_1\\
&\qquad - \left(\boldsymbol{\Delta}_1\boldsymbol{\Xi}_1 + \boldsymbol{\Delta}_2\boldsymbol{\tau}^T\boldsymbol{S}_0\boldsymbol{V}_\times\right) d\boldsymbol{Q}_2\Bigg) && \text{(47)}\\
&= \Bigg((\eta\boldsymbol{I}_3 - \eta\left(1 + \frac{\eta\boldsymbol{V}^T\boldsymbol{N}}{\sqrt{\xi}}\right)\boldsymbol{N}\boldsymbol{N}^T V\\
&\qquad + \boldsymbol{H}\boldsymbol{\Delta}_1\left(\boldsymbol{\tau}^T\boldsymbol{S}_1\boldsymbol{E}_{0\times} - \boldsymbol{\gamma}^T\boldsymbol{S}_1\boldsymbol{T}_\times\right) + \boldsymbol{H}\boldsymbol{\Delta}_2\left(\boldsymbol{\tau}^T\boldsymbol{S}_0\boldsymbol{E}_{1\times} - \boldsymbol{\zeta}^T\boldsymbol{S}_0\boldsymbol{T}_\times\right)\Bigg) d\boldsymbol{V}\\
&\qquad + \boldsymbol{H}\lambda_0 d\boldsymbol{N}_0 + \boldsymbol{H}\lambda_1 d\boldsymbol{N}_1 + \boldsymbol{H}\lambda_2 d\boldsymbol{N}_2\\
&\qquad + \boldsymbol{H}\left(\boldsymbol{\Delta}_1\boldsymbol{\gamma}^T\boldsymbol{S}_1 + \boldsymbol{\Delta}_2\boldsymbol{\zeta}^T\boldsymbol{S}_0\right)\boldsymbol{V}_\times d\boldsymbol{P}\\
&\qquad + \boldsymbol{H}\left(\boldsymbol{\Delta}_1\left(\boldsymbol{\tau}^T - \boldsymbol{\gamma}^T\right)\boldsymbol{S}_1\boldsymbol{V}_\times + \boldsymbol{\Delta}_2\left(\boldsymbol{\tau}^T - \boldsymbol{\zeta}^T\right)\boldsymbol{S}_0\boldsymbol{V}_\times + \boldsymbol{\Delta}_2\boldsymbol{\Xi}_0 + \boldsymbol{\Delta}_1\boldsymbol{\Xi}_1\right) d\boldsymbol{Q}_0\\
&\qquad - \boldsymbol{H}\left(\boldsymbol{\Delta}_2\boldsymbol{\Xi}_0 + \boldsymbol{\Delta}_1\boldsymbol{\tau}^T\boldsymbol{S}_1\boldsymbol{V}_\times\right) d\boldsymbol{Q}_1\\
&\qquad - \boldsymbol{H}\left(\boldsymbol{\Delta}_1\boldsymbol{\Xi}_1 + \boldsymbol{\Delta}_2\boldsymbol{\tau}^T\boldsymbol{S}_0\boldsymbol{V}_\times\right) d\boldsymbol{Q}_2, && \text{(48)}
\end{aligned}
$$

where $\boldsymbol{H} = -\left(\mu\boldsymbol{I}_3 + \eta\left(1 + \frac{\eta\boldsymbol{V}^T\boldsymbol{N}}{\sqrt{\xi}}\right)\boldsymbol{N}\boldsymbol{V}^T\right)\boldsymbol{J}$.

4 Conclusion

In this work we have evaluated the full first order spatiotemporal structure of light's interaction with dielectric materials as reflection and refraction. In contrast to earlier work, [Igehy, 1999, Schjøth et al., 2007], we make only assume that the geometry is given as a piecewise smooth surface. The derivation allows for easy extension to other parameters than viewing directions, and parallel rays are briefly treated as a special case as well as velocities on both the view point, direction and surfaces. Finally, we give two examples of common shading models, flat and Phong, in full detail.

Conceptually, we model ray bundles instead of rays and obvious applications are ray tracing and photon splatting, but the methodology is naturally and easily extended to all phenomena well approximated by first order Taylor series. Our generalization offer more accurate and faithful reconstruction of ray bundles in space and time.

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