METRIC GEOMETRY, NON-POSITIVE CURVATURE AND COMPLEXES

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. These notes give mathematical background on CAT(0) spaces and related geometry. They have been prepared for use by students on the International PhD course in Non-linear Statistics, Copenhagen, June 2017.

1. Metric geometry and the $\mathrm{CAT}(0)$ condition

Let X be a set. A metric on X is a map $d : X \times X \to \mathbb{R}_{\geq 0}$ such that

(1) $d(x, y) = d(y, x)\ \forall x, y \in X$,
(2) $d(x, y) = 0 \Leftrightarrow x = y$, and
(3) $d(x, z) \leq d(x, y) + d(y, z)\ \forall x, y, z \in X$.

On its own, a metric does not give enough structure to enable us to do useful statistics on X: for example, consider any set X equipped with the metric $d(x, y) = 1$ whenever $x \neq y$.

A geodesic path between $x, y \in X$ is a map $\gamma : [0, \ell] \subset \mathbb{R} \to X$ such that

(1) $\gamma(0) = x$, $\gamma(\ell) = y$, and
(2) $d(\gamma(t), \gamma(t')) = |t - t'|\ \forall t, t' \in [0, \ell]$.

We will use the notation $\Gamma(x, y) \subset X$ to be the image of the path γ and call this a geodesic segment (or just geodesic for short).

A path $\gamma : [0, \ell] \subset \mathbb{R} \to X$ is locally geodesic if there exists $\epsilon > 0$ such that property (2) holds whever $|t - t'| < \epsilon$. This is a weaker condition than being a geodesic path.

The set X is called a geodesic metric space if there is (at least one) geodesic path between every pair of points in X. It is uniquely geodesic if there is exactly one geodesic between every pair of points.

Suppose $c : [a, b] \to X$ is a path. The length of c is

$$\ell(c) = \sup_{a=t_0 \leq t_1 \leq \cdots \leq t_n = b} \sum_{i=0}^{n-1} d(c(t_i), c(t_{i+1}))$$

where the supremum is taken over all possible n and partitions of the interval $[a, b]$. The length of c is infinite when this expression is unbounded.

A metric space is a length space if $d(x, y)$ is the infimum of lengths of paths connecting x, y for all $x, y \in X$. (Note: this is how the metric on tree-space will be constructed.) When is there a path that realizes the infimum? The Hopf-Rinow theorem states that any complete locally compact length space X is a geodesic metric space.

Example: $\mathbb{R}^2$ with the origin removed is a length space but not a geodesic metric space, since there is no geodesic between antipodal points.

There is a notion of curvature in a geodesic metric space X: the idea is to look at whether triangles are 'fat' or 'thin'. Given $p, q, r \in X$ the geodesic triangle $\Delta(p, q, r)$ is a choice of geodesics $\Gamma(p, q), \Gamma(q, r), \Gamma(r, p)$. A triangle $\Delta' = \Delta(p', q', r')$ in $\mathbb{R}^2$ is a comparison triangle if

$$d(p, q) = d(p', q'), \quad d(q, r) = d(q', r') \quad \text{and} \quad d(r, p) = d(r', p').$$

Such a triangle always exists in $\mathbb{R}^2$ (because Δ satisfies the triangle inequality in X) and is unique up to isometries of $\mathbb{R}^2$.

Given $x \in X$ on $\Gamma(p, q)$, a comparison point x' in Δ' is a point on $\Gamma(p', q')$ such that

$$d(x, p) = d(x', p') \quad \text{and} \quad d(x, q) = d(x', q').$$

The triangle $\Delta(p, q, r)$ satisfies the CAT(0) inequality if $d(x, r) \leq d(x', r')$ for all pairs (x, x') and similarly for the other two edges of the triangle. The geodesic metric space X is a CAT(0) space is every geodesic triangle satisfies the CAT(0) inequality.

CAT(0) spaces are often called non-positively curved spaces. There is a very similar definition of a CAT(κ) space for $\kappa \neq 0$. These spaces can be thought of as having curvature $\leq \kappa$. The definition is the same as for CAT(0) spaces, except that comparison triangles are constructed in a model space M_κ. For $\kappa < 0$, M_κ is a scaled version of the hyperbolic plane; for $\kappa > 0$, M_κ is a scaled version of the sphere S^2. In order to be CAT(κ) triangles must be 'thinner' than those on the model space.

In contrast, there is a definition of non-negatively curved geodesic spaces, due to Alexandrov: X is non-negatively curved if every geodesic triangle in the space is at least as 'fat' as a comparison triangle in $\mathbb{R}^2$.

Example: the k-spider is CAT(0). More generally, metric trees are also CAT(0).

Theorem: A smooth Riemmannian manifold M is locally CAT(κ) if and only if the sectional curvature of M is $\leq \kappa$.

2. Properties of CAT(0) spaces

Suppose X is a CAT(0) space.

2.1. Geodesics.

(1) There is a unique geodesic $\Gamma(x, y)$ for all $x, y \in X$.
(2) $\Gamma(x, y)$ varies continuously with its end points.
(3) Every local geodesic is a geodesic.

2.2. Convexity and projection. A function $f : I \to \mathbb{R}$ defined on an interval $I \subset \mathbb{R}$ is convex if

$$f((1-s)t + st') \leq (1-s)f(t) + sf(t')$$

for all $t, t' \in I$ and $s \in [0, 1]$.

A function $f : X \to \mathbb{R}$ is convex if for any geodesic path $\gamma : I \to X$ parametrized proportional to length, the function $I \to \mathbb{R}$ defined by $t \mapsto f(\gamma(t)))$ is convex.

Given $x \in X$, it can be shown that the distance function $d(\cdot, x) : X \to \mathbb{R}$ is convex. Similarly the function $d(\cdot, \cdot)$ is convex on the product space $X \times X$.

A subset $A \subseteq X$ is convex if $\Gamma(x, y) \subseteq A$ for all $x, y \in A$.

Suppose that $A \subseteq X$ is convex and complete in the induced metric. (Tree-space is complete, so any closed convex subset satisfies these criteria.) Then given any $x \in X$ there is a unique point $\pi(x) \in A$ closest to x:

$$\pi(x) = \arg\min_{a \in A} d(a, x).$$

This is called the projection of x onto A.

3. CUBICAL COMPLEXES

Cubical complexes are a rich source of examples of CAT(0) spaces, and include tree-space. A condition due to Gromov specifies when a cubical complex is CAT(0) and it can be checked quite readily in examples. We describe Gromov's condition in this section.

Let $I^n \subset \mathbb{R}^n$ be the unit cube $[0, 1]^n$ equipped with the Euclidean metric. The codimension-k face of I^n corresponds to fixing k coordinates on I^n to be either 0 or 1. A cubical complex is a metric space obtained by gluing together cubes (potentially of different dimensions) along their faces: a codimension-k face in one cube can be glued isometrically to one or more codimension-k faces in other cubes. The following constraints must be satisfied:

(1) Any two cubes are connected by at most one identification.
(2) No cube is glued to itself.

A cubical complex X can be given a metric as follows. The distance between $x, y \in X$ is defined to be the infimum of the lengths of paths between x and y which are straight line segments within each cube. A 'picture proof' shows that this distance satisfies the triangle inequality. It follows that X is a length space. When a cubical complex is locally finite then it is in fact a geodesic metric space (Hopf-Rinow theorem).

We will need some other types of complex.

(1) **Metric simplicial complexes.** An n-dimensional metric simplex is the convex hull of $n + 1$ points in $\mathbb{R}^n$. A metric simplicial complex is formed by gluing together a collection of simplices along their faces via isometries. The intersection of two simplices is either empty or a shared face of the simplices.
(2) **Spherical complexes.** The component parts of these complexes are subsets of spheres defined as $\text{Sph}_n = S^n \cap R^{n+1}_{\geq 0}$. The faces of Sph_n are isomorphic to some Sph_k with $k < n$.
(3) **Abstract simplicial complexes.** Let V be a set. An abstract simplicial complex is a collection of subsets of V with the property that if $A \subseteq V$ is in the collection then so is every subset $B \subseteq A$.

A complex C is called flag if

$$\text{a } k\text{-simplex is in } C \quad \Leftrightarrow \quad \text{all its 1-dimensional faces are in } C$$

The link of a vertex v in a cubical complex X is a spherical complex defined by

$$L(v) = x \in X : d(x, v) = \epsilon$$

where $0 < \epsilon < 1$ is a fixed constant.

The Gromov condition states that a cubical complex X is CAT(0) if and only if the link of every vertex is flag.

Example: $C_{k\pi/2}$ can be realized as a cubical complex by gluing infinitely many squares together to form each quadrant $\mathbb{R}^2_{\geq 0}$. Then $C_{k\pi/2}$ is CAT(0) when $k \geq 4$, but not when $k = 3$.

4. Billera-Holmes-Vogtmann tree-space and orthant spaces

$\mathcal{T}_N$ is CAT(0):

- Construct each orthant from an infinite collection of cubes.
- Each cube (of dimension $N - 3$ or lower) corresponds to a pairwise compatible set of splits.
- Each spherical face of a link is present if and only if it corresponds to a set of compatible splits i.e. if and only if its 1-dimensional faces are present.
- $L(v)$ is therefore flag for all vertices v.

Orthant spaces are a generalization of BHV tree-space:

- Fix an abstract simplicial complex K on a vertex set V.
- Suppose K is flag: $F \in K \Leftrightarrow$ every set $\{v_1, v_2\} \subseteq F$ is in K.
- Associate every $v \in V$ with an axis in $\mathbb{R}^{|V|}$.
- Every $F \in K$ determines a positive orthant in $\mathbb{R}^{|V|}$.
- The orthant space associated to K, V is the union of these orthants.
- Gromov condition $\Rightarrow$ CAT(0).

Associate a graph to K, V: one vertex for each $v \in V$ and an edge between v_1, v_2 if $\{v_1, v_2\} \in K$. The orthants correspond to clique sets in this graph.

Statistical interpretation: suppose potential individuals / objects in a study exhibit various continuous traits we can measure. Also suppose there is a rule which says whether any pair of traits (X, Y) can be displayed simultaneously by an individual (compatible traits) or not (incompatible traits). Assume that a value of zero for a trait is equivalent to the individual not displaying the trait. A data set consisting of some individuals, the traits they display and the measured values of these traits corresponds to points in an orthant space.

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